Taxicab Geometry, or When a Circle is a Square

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Abstract

The distance between two points is the length of the shortest path connecting them. In plane Euclidean geometry such a path is along the straight line connecting the two points. In contrast, in a city consisting of a square grid of streets shortest paths between two points are no longer straight lines (as every cab driver knows). We will explore the geometry of this unusual distance and play several related games.

Game One: dispatch the firetruck

An accident takes place at some intersection in the city. There are two police cars nearby. Decide which car should be dispatched to the accident site depending on the coordinates of the accident and the current positions of the police cars. Remember that cars can only drive along the vertical or the horizontal streets in the city. We also assume that both cars travel with the same speed.

Start by drawing the routes the cars will be taking.

1. Accident site: $(5, 0)$.
   First car: $(2, 0)$.
   Second car: $(9, 0)$.

2. Accident site: $(1, 1)$.
   First car: $(1, 5)$.
   Second car: $(6, 1)$.

3. Accident site: $(2, 3)$.
   First car: $(2, 0)$.
   Second car: $(9, 0)$.

The Taxicab Distance

Let $A = (a, b)$ and $B = (c, d)$ be two points on the coordinate plane. In the usual plane geometry, the shortest route between these two points is along the straight line connecting them. The distance between these two points is

$$d(A, B) = \sqrt{(a - b)^2 + (c - d)^2}.$$

For example, the distance between the points $(0, 3)$ and $(4, 0)$ is $\sqrt{3^2 + 4^2} = 5$.

Imagine that you are only allowed to move along vertical lines and along horizontal lines. (Such is the case in a city which only has streets running in the north-south direction and in the east-west direction). So, let's call such a route a *taxi route.*

Problem 1.

1. Draw the shortest possible *taxi route* from point $A = (0, 3)$ to point $B = (4, 0)$.

2. Find the length of this route.

3. Is there another taxi route that also gives you the shortest possible distance between the two points? Draw as many shortest routes as you can (use different colors).

We will call the distance between points A and B obtained by going along one of the shortest taxi routes the *taxicab distance.*

Problem 2.

Find the *taxicab distances* between the following points:

1. $(1, 0)$ and $(1, 7)$.

2. $(3, 2)$ and $(5, 2)$.

3. $(4, 3)$ and $(12, 1)$.

4. Can you describe how the taxicab distance is computed in words?

For points $A = (a, b)$ and $B = (c, d)$, the *taxicab distance* is given by

$$d_{\text{taxi}}(A, B) = |a - c| + |b - d|,$$

where $|a - b|$ denotes the absolute value of the difference between a and c, and $|b - d|$ denotes the absolute value of the difference between b and d.

Problem 3. Let's compare the usual Euclidean distance and the taxicab distance.

1. Give an example of two points such that the Euclidean (usual) distance and the taxicab distances between them are equal to each other.

2. Give an example of two points such that the taxicab distance is bigger than the Euclidean distance.

3. Can you find a pair of points for which the Euclidean distance is bigger? Why?

Problem. Consider the line segment joining points $(2, 0)$ and $(0, 2)$ on the plane.

1. What is the point on this segment that is closest to $(0, 0)$ in the usual sense (with respect to the u Euclidean distance)?

   - Find the distance from this point to $(0, 0)$;
   - Find the taxicab distance from this point to $(0, 0)$;

2. Write down the equation of the line through $(2, 0)$ and $(0, 2)$.

3. Let (x, y) be a point on the segment between $(0, 2)$ and $(2, 0)$. What is the taxicab distance from (x, y) to $(0, 0)$? How does it depend on (x, y)?

4. What is the point (or points) on this segment that are closes to $(0, 0)$ with respect to the taxicab distance?

One of the properties of straight lines in Euclidean geometry is that the distance between any two points is along a straight line. In taxicab geometry, straight lines in general (unless they are vertical or horizontal) no longer play this special role.

Circles and π in Taxicab Geometry

In Euclidean plane geometry, a *circle* is usually defined as the set of all points which are at a fixed distance from a given point. The given point is the *center* of the circle. The fixed distance is the *radius* of the circle. *Diameter* is the longest possible distance between two points on the circle and equals twice the radius. *Circumference* is the length of the circle.

The equation of the (Euclidean) circle of radius r centered at point with coordinates (a, b) is

$$(x - a)^2 + (y - b)^2 = r^2.$$

This follows from the formula for the distance.

The same definitions of the circle, radius, diameter and circumference make sense in the taxicab geometry (using the taxicab distance, of course). However, taxicab circles look very different.

Problem.

1. The taxicab circle centered at the point $(0, 0)$ of radius 2 is the set of all points for which the taxicab distance to $(0, 0)$ equals to 2. Draw the taxicab circle centered at $(0, 0)$ with radius 2.
    - What is the shape of this taxicab circle?
    - Do you think other taxicab circles will have the same shape?
2. Draw the taxicab circle centered at the point $(4, 0)$ of radius 4.
3. Find the points of intersection of these two circles.

Problem.

1. Give an example of two taxicab circles which have exactly one common point.

2. Give an example of two taxicab circles which have more than one common point.

In Euclidean geometry, π is defined as the ratio of the circumference to the diameter:

$$\pi = \frac{\text{Circumference}}{\text{Diameter}} = \frac{\text{Circumference}}{2 \times \text{Radius}} \simeq 3.14$$

Problem. What is the value of π (the ratio of the circumference to the diameter) in taxicab geometry? To find out, consider a taxicab circle with diameter d centered at $(0, 0)$. Find the taxicab circumference of this circle. Then, compute the ratio

$$\pi_{\text{taxi}} = \frac{\text{Taxicab Circumference}}{\text{Taxicab Diameter}} =$$

Compare it with the value of π in the usual Euclidean geometry.

The equation for the taxicab circle centered at (a, b) and having radius r is

$$|x - a| + |y - b| = r.$$

Game Two: Find the Hidden Treasure

We will play the following game:

Player I (instructor) "hides" the treasure at a certain intersection in the City of Descartes with taxicab distance. (The coordinates of the point are written on a piece of paper and are hidden).

Player II (students) choose a point and asks Player I about the distance from this point to the treasure.

We will play this game several times with the goal of finding out the smallest number of questions Player II needs to find the treasure.

Perpendicular bisectors in Taxicab Geometry

1. Let A and B be points on the Euclidean plane. Describe the set of all points which are at the same distance from A as they are from B. Draw how this set looks like for several examples of positions of A and B below.

2. Find the set of all points which are at the same taxicab distance from A and B in the following cases. Draw the set on the plane:

   (a) $A = (1, 0)$ and $B = (9, 0)$;

   (b) $A = (3, 3)$ and $B = (8, 8)$;

   (c) $A = (0, 0)$ and $B = (4, 2)$;

   (d) $A = (0, 4)$ and $B = (4, 2)$;

Application: School Districts' Boundaries

Lloyd's game: Catch the Rooster and the Chicken