Any Conic Section (So Long as It Is a Parabola)

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Imagine a very thirsty and hungry fly flying above a plane. It notices a river (an absolutely straight line) and a pile of food (of course, it's a point off the line which is the river) and tries to land. Unfortunately, it's a totally logical creature and it's equally attracted to water and food. After an anguishing moment of indecision it lands at a spot which is at the exact same distance from both attractors. But alas, the fly cannot reach either of them! So instead of sitting still it attempts to move. Sure thing, it remains equidistant from both the river and the food pile at every spot of its path.

1. (a) Draw the poor fly's path. Use the grid paper and devise a method which produces points at exactly the same distances from one chosen (horizontal) line and a chosen (lattice) point above this line.
   (b) Explain your method to your neighbor(s).

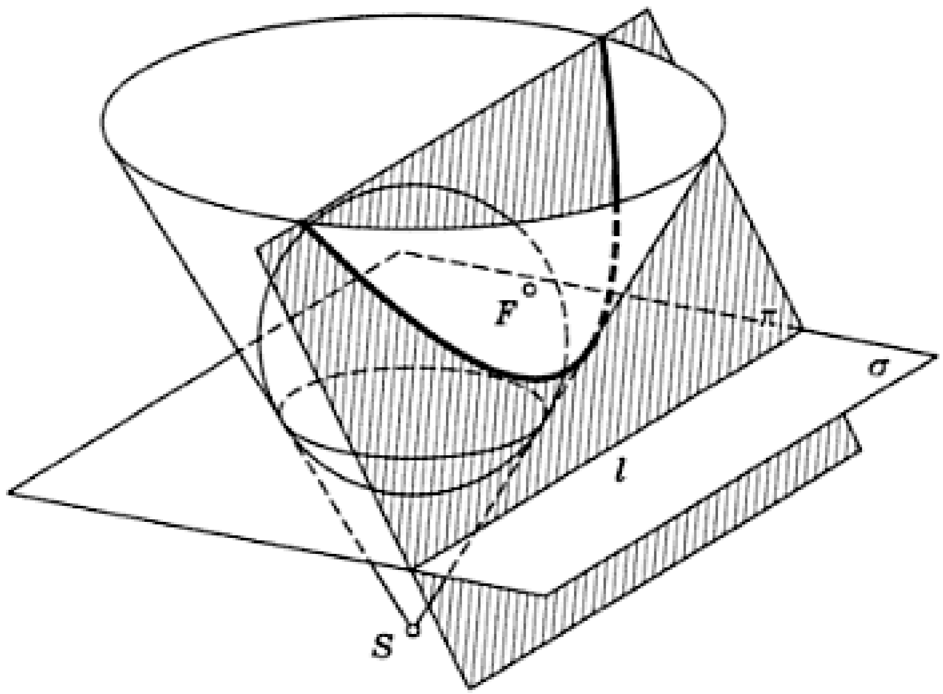
2. Given a point on the fly's path, how do you find the tangent line at that point?

The fly's path described above – the set of all points in the plane which are equidistant from a given line in the plane and a fixed point off this line – is called a *parabola*. The fixed point is called a *focus* and the line is called the *directrix*. The line through the focus and perpendicular to the direcrix is the *axis* of the parabola (and of course, it's its axis of symmetry).

3. What happens with a light ray which enters a parabola parallel to its axis and then bounces off the parabola?

An ellipse and hyperbola have a number of similar descriptions – either using foci and directrices, or only foci. But everybody knows that their collective name is *conic sections*, and rightly so since they can be obtained by intersecting a doubly infinite cone with variously tilted planes. It's an interesting question to ask – how do we know that the curves defined by means of plane distances and the curves obtained by slicing a 3-d cone are indeed the same creatures?

The easiest way to answer this question (well, at least the easiest that I know of) is by using Dandelin spheres (see [1] or [2]). Do it when you have time – it's delightful! For the case of a parabola the diagram below (borrowed from [1]) would be helpful:



So far we've been talking about lengths; at some point (no pun intended!) it might get a bit boring. So how about switching to a two-dimensional measure, the area? To those of us who find social issues appealing, let's delve into a question of equality, or, rather, equal sharing.

4. Dana brings to school a piece of bread with a piece of salami on top of it, and wants to split it evenly with a friend. Is it possible to cut it with a knife just once so that each of two parts has exactly the same amount of bread and salami? (Of course, we assume that a piece of bread is a rectangle, a piece of salami is a circle situated anywhere on the bread, and a 'single knife cut' means intersecting these rectangle and circle by a single straight line.)

5. Is it possible to split a triangle into two parts such that:

a) they have the same area;
b) the area of one part is twice the area of the other one;
c) the area of one part is twice the area of the other one, and the sides of the original triangle are intact?

To connect this last problem with conic sections, let's consider a special kind of a triangle, called *Archimedes triangle.* This is a triangle formed by a secant of a parabola and two tangents to the parabola at the end points of the secant.

6. Is every triangle an Archimedes triangle?

The region inside an Archimedes triangle bounded by the secant and the parabola is called a *parabolic segment.*

7. What is the area of a parabolic segment compared to the area of the corresponding Archimedes triangle?

This problem 7 was first solved by Archimedes about 19 hundred years before calculus and about 21 hundred years before fractal geometry!

The final problem on this list is not about area, but it's too good to skip:

8. (From [1])
Two travelers move along two straight roads with constant speed. Prove that the line connecting them is always tangent to some parabola. (The roads are not parallel and travelers pass the intersection at different times.)

Further reading

1. *Geometry of Conics*, by A. V. Akopyan and A. A. Zaslavsky, AMS, Mathematical World, vol.26

2. *Conics and Dandelin spheres*, by Rita Kós,
http://www.komal.hu/cikkek/dandelin/dandelin.e.shtml

3. *Cut the Knot*, by Alexander Bogomolny,
http://www.cut-the-knot.org/ctk/Parabola.shtml