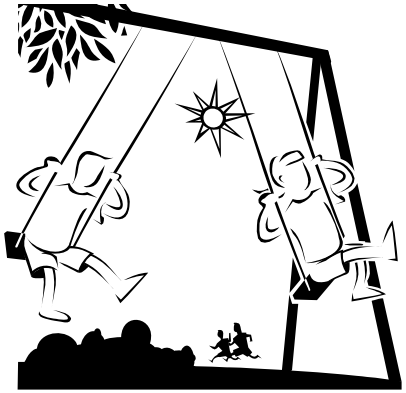

Math Circle as a Problem Solving Playground

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About me

- “When there is no tiger in the mountains, a monkey becomes the king” – Chinese proverb

- Mom of 3 young kids, system engineer at NASA
- Teaching 2 groups of elementary school students (6-9 yrs old) weekly for 2 years.

“System engineering” approach

- What is the current situation?
- What are we trying to accomplish?
- How do we define the success of the “mission”?
- How do we define the minimal success of the “mission”?
- What our resources are?

ELEMENTARY SCHOOL MATH – the problem we face

- The elementary school curriculum's focus is on number manipulation
- The kids are conditioned to " sit and listen"
- Textbooks, (if any) are dry and patchy.

- *" In school, we just convert some numbers into new numbers.. and it is so-o boring!" – Ana L., 8 yrs old*

..but there is still hope:

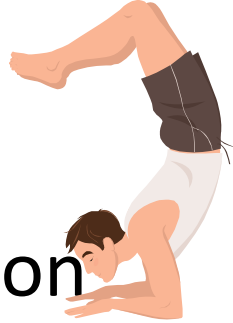
- *“I hate math, but I love thinking!”*

  –Lilien M., 9 yrs old.

POWER AND FLEXIBILITY OF MIND

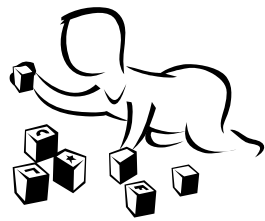
- *"It is our choices…that show what we truly are, far more than our abilities." Harry Potter and the Chamber of Secrets*

POWER AND FLEXIBILITY OF MIND



- **Power** - technique and speed; ability to move on
- **Flexibility** -ability to disengage, choose and switch

- These two qualities seem to be independent. To be a good problem solver, a person needs both. However, the later one is almost never taught in school, and barely in college.
- In order to learn the **flexibility of mind**, kids need to learn how *play with the problem* - without the pressure to get a specific result by specific time. They need to *make mistakes and learn to enjoy them*!



LACK OF MATERIALS FOR PARENTS AND STUDENTS

- ..and we are not talking Kumon here ☺
- Very few school materials that teach the flexibility of mind
- Very few materials that teach math from the historical point of view, presenting the challenges faced by the explorers

IDEAL TEXTBOOK

- “Dear reader and fellow explorer,
- Today I would like to tell you about the problem I was trying to solve, my motivation to do so, my initial expectations, and the evolution of my understanding.
- First, I tried to do this.., and failed miserably. Analyzing my errors, I realized that..
- Now, I am at the point when.. My understanding is not perfect, but looking backwards I can give you a couple of advices..”

HUMAN VOICE BEHIND THE TEXT

- Most of the current math textbooks (especially the ones adopted by school districts) lack the human voice behind the text.
- The student does not feel that the author cares about the reader, or that the author is human at all. They look like they are written to be read by computers. And children are sensitive to that.
- The school textbooks are written by technical writers, not by the best minds in the field.
- Math, the way it is presented in school and in textbooks, seems boring and pointless even to the brightest kids.

TEACHERS'S ROLE

- Teacher is a role model for problem solving. In order to teach thinking skills, the teacher has to demonstrate the problem solving process to the class, to make mistakes and fix them, to fail and to recover, and always adjust to the class.
- There are very few teachers who can afford this while "teaching to the test".

ROLE OF THE ELEMENTARY MATH CIRCLE:

- Show that math is fun
- Provide the place to play with math
- No grades/no time pressure
- Go “deeper”, not “ faster”
- Show the “human face” of math
- Focus on problem solving skills

PROBLEM SOLVING IN ELEMENTARY MATH CIRCLE

MATH CIRCLE AND THE FREEDOM OF ARGUMENT

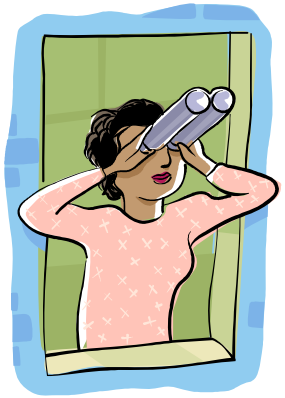
- Fear can completely block any mental activity. The children should be encouraged to freely (but politely! ☺) argue with their peers and the teacher, without being reprimanded or feeling stupid.
- Demonstrate your thinking process. Feel free to make silly mistakes in front of your students, let them find your logical fallacies, and let them hear you thinking aloud.

PROBLEM SOLVING TECHNIQUES

- Here is a list of some techniques I tried to introduce in class:
- Self-awareness (unstated assumptions, dealing with uncertainty)
- Exploration of multiple solutions/domains
- Step- by-step thinking (intro to logic, simplification, proofs)
- Symmetry tests
- Solving backwards
- Educated guess/estimation
- Ideal result/extreme cases
- Sanity checks

ATTEMPTS AND OBSERVATIONS

- Today, I am going to share my efforts in teaching some problem solving skills to elementary school students, along with my thoughts and observations. I hope to give you enough material to try <u>your</u> problem solving skills on ☺



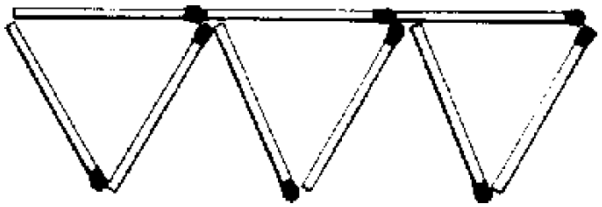
- “Math is like a slippery fish” –
  Noah Z., 7 yrs old

SELF-AWARENESS

- ***"What most experimenters take <u>for granted</u> before they begin their experiments is infinitely more interesting than any results to which their experiments lead". Norbert Wiener***
- The best way to test for one's unstated assumptions is to try some "insight" or " out-of the box" problems. These are the problems that exploit the biases of our mind, so they seem insolvable at first– but when you know the solution, it seems obvious.
- The best hint to give the students: " Is there something you take for granted?"

Example

- We speak about unstated assumptions on every lesson. Consider this example:
- Move 3 matches to get 5 triangles.



- The fist response is: “ It is impossible!”
- Kids continue to move the matches. They are able to get 4 equilateral triangles (several solutions),but not 5
- Teacher: “ Take a look on your designs. Are you trying to hold on to something you do not need?
- Student 1:” Oh! They may be of a different sizes!”
- Student 2: “ And they may be inside one another!”

THE POWER OF INSIGHT PROBLEMS

- ***"To see meaning and understanding come where there has been no meaning and no understanding is to share the work of a demiurge." Norbert Wiener***
- "Insight" problem solving may be the single most important skill you introduce to the students – and it creates a very powerful feeling of achievement, something they will not easily forget.

INSIGHT PROBLEMS - CONTINUED

- Insight problems also motivate kids to look for "**insightful" ways of solving "regular" problems**, approaching them from different angles, recognizing and avoiding clichés.
- Kids need to learn to **search for and to recognize** the feeling of the "insight" moments; to distinguish between knowledge and understanding.
- My observations show that **if kids expect an " insightful" solution, they are much more likely to find one.**
- From the teacher's perspective, there is a limited number of insight problems literature, suitable for the little children – that is, hopefully, until you learn to recognize those insight moments in your and other's life, and **compose these problems yourself.**

UNSTATED ASSUMPTIONS

- ***" .. in prejudice there is a great deal of comfort, a great deal of pleasure" J. Krishnamurti***

- The insight problems require the solver to **critically** review his/her assumptions .
- Over the time, I see my students learning to **stop and re-assess** their assumptions even without this hint from me.
- Besides the practical problem solving skills, it teaches the kids to have an open mind. It also makes them more tolerant to alternative solution approaches.

OPEN-ENDED PROBLEMS

- *“Good morning, said Bilbo. -What do you mean? Do you wish me a good morning, or mean that it is a good morning whether I want it or not; or that you feel good on this morning; or that it is a morning to be good on?” J.R.R. Tolkien*

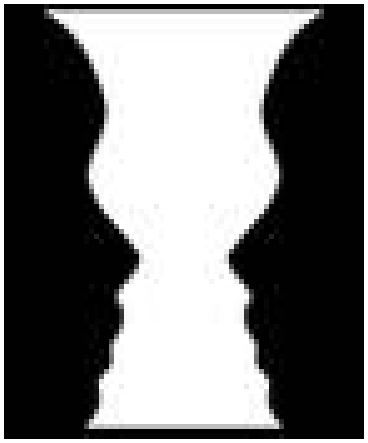
OPEN-ENDED PROBLEMS

- Most of the problems we experience in life are **open-ended problems**. They have a **set of solutions**, which are all possible and viable, though some may be more effective than others.
- In school, kids are conditioned to look for just " **one right answer**". When they come to math circle, it takes some time for them to accept that there are many (sometimes infinitely many)possible solutions.
- These problems are very relaxing to the kids, as they can let their imagination go wild. They are good for class warm-ups or end of the class.

FOR INTRODUCTION: OPTICAL ILLUSIONS

- In order to educate the kids about unstated assumptions, we spend time exploring optical illusions and other “weird” things.
- They are also a great way to initiate a discussion about the problems with multiple solutions.

Explore the picture. What do you see?



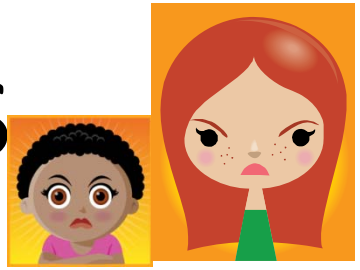
Is there only one right answer to this question?
Hint for the students: Put aside your first solution and try to come up with a very different one.

SOURCES OF OPEN-ENDED PROBLEMS

- Open-ended problems are easy for the teacher to come up with. You can suggest the problems that **originate in history, fairy tales or daily life**, which makes them even more interesting to the kids.

- Roman emperor Tiberius was very interested in his future. He would invite an astrologist, ask him for a prediction, and then push the astrologist off the cliff, so nobody would learn the emperor's fate. However, once an astrologist came to the emperor, but survived– how did he do that?

ANALYZING SOLUTIONS



- One kid can annoy one mom a day. How long will it take 10 kids to annoy 10 moms?
- Anton: Ten days, if they are taking turns (one day one kid annoys his mom; another day another kid annoys his, and so on..)
- Marie: One day, if they are angry at their moms, who did not allow them to have a play date together that day!
- Imran: One hundred days, if first one kid annoys all ten moms, one mom a day; then another one does the same, and so on…
- (At this point, I vividly imagined this situation..)

ANALYZING SOLUTIONS

- A problem from ancient Egypt: **Divide 7 round breads evenly among 8 people.**
- A modern student instantly comes up with the solution of 7/8 – that is, divide each bread into 8 pieces, and then give 7 pieces to each person. This solution is an acceptable one, however, it requires 28 cuts. I asked if there is a more effective solution (give each person ½, ¼ and 1/8 of the bread). As you may calculate, this way is much more effective – just 12 cuts. (If you consider a similar problem in industry , this approach may save thousands of dollars).
- Besides obvious excitement that is generated by these types of problems, children explore the set of possibilities and learn how to **range the solutions in terms of effectiveness and beauty.**

PROBLEMS WITH NO SOLUTIONS

- While the children thoughtfully enjoy the open-ended problems providing for many solutions, it is very interesting to notice how **uncomfortable** it is to the children to think of " no solution" solution. They will resist this idea until they are completely stuck.
- While I would not like to teach " learned helplessness" to the kids, a small amount of these problems is definitely healthy. I would like to see them to recognize that sometimes even the best efforts are in vain, and learn to deal with that.
- Again, no such problems in school.

Example

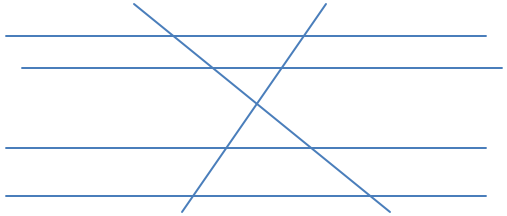
- **There are 4 glasses on the table – 3 are upright, and 1 is upside-down. You may turn over 2 glasses at a time. Is it possible to get all glasses upright?**
- The problem took half of the class time (children never want to give up). Boys (Ben and Anton) decided to turn just one glass, but twice, (up and down) and claimed it a solution. When asked to show us how they do it, they weren't able to reproduce the claimed result, however, they were extremely proud of their " out of the box" thinking nevertheless.
- (Ben:" No I did it!" Ana" No, you did not!" Dana asks Ben to demonstrate. Ben" Oh.. but I have it another way!")
- Vicky and Marie were able to actually formulate their statement (with some student assistant' s help) – they explained that adding or subtracting 2 from the odd number does not change it to even.

ILL-DEFINED PROBLEMS

- Related to the problems with no solutions are **ill-defined problems** that I (sometimes with no intention to do so) give to the kids from time to time. These problems force the students to try various strategies, modify their plans and re-evaluate their goals along the way. Eventually, they learn to recognize an ill-defined problem and re-state it . (There is another opportunity for them to see the teacher as a human being, making mistakes..)
- Good news - these types of problems are <u>easy to come up with</u> for the teacher! ☺

Example

- Could you build 4 triangles with just 6 toothpicks?
- This one was definitely an ill-defined problem, and I had to continue to impose limitations as we went on. First, I asked them to make 4 triangles with 6 toothpicks. Noah broke the toothpicks in half, and built those. Marie used the toothpicks so they will not touch, (built a parallelogram with diagonals) and just indicate the direction. I had to explain that they have to use a non-broken toothpick. I also told them that the triangles need to be equilateral. Robert quickly built a set of 4 equilateral but unequal triangles (see below)



- Finally, I told them they need <u>4 equal equilateral triangles</u>. That confused the kids for a minute, and Lilien asked for a hint. I reminded them to look whether they are limiting themselves to something not stated in the problem. “2D!” exclaimed Lilien. “We are limiting ourselves to a flat surface!”

LOGICAL PARADOXES

- *"It is bad luck to be superstitious."*

LOGIC PARADOXES

- When I was a child, I was told a story. “A child is playing piano scales. The dad is reading a newspaper. Some kids are playing soccer in front of their windows. “ Daddy, may I go play?” whines the child. “OK”, answers the dad. The son jumps off the chair and runs out. The dad gets up, still looking into the newspaper, comes to the piano and finishes the musical scale.”
- The dad in this story was not able to stand an unfinished musical scale. The same is true with the logic paradoxes - it is very difficult for us humans to stand something as unfinished.
- Logic paradoxes are a **specific set of open –ended problems**. The best thing I like about the logic paradoxes that they **instantly make you think – and constantly question your assumptions.**

STEP-TO-STEP THINKING

- Logical thinking
- Simplification
- Proofs

LOGICAL THINKING

- Kids have a natural ability to think logically. You do not have to teach them that. What they do not know is a good notation – and this should be taught, demonstrating the best and worst practices, and identifying the possible slips.

SIMPLIFICATION

- Finding a similar but an easier problem is very challenging to the kids of this age.
- They need a lot of prompting and coaching.

- Question: Should we teach them this or should we just wait? (get back to me with your thoughts and observations)

PROOFS

- Do not attempt to teach formal proofs in elementary school circle – you are going to fail.. ☹
- Do introduce the idea of proofs – informally.
- Consider their knowledge base.
- Make jokes about proofs.

PROOF DISCUSSION - 1

- Teacher:” Prove that clouds are closer than the Sun”
- Student, 6 yrs old: “ Do you mean during the day or during the night?”
- Student, 7 yrs old:” I do not know, but we can send a robot to check..”
- Student, 8 yrs old:” An airplane can fly through the clouds, but Sun is still above..”

AN OFF-TOPIC OBSERVATION

- At that point, I showed them a pen and a book, and put a pen in front of the book. What is closer – the pen or the book?
- Vicky says that the pen is closer to them, since the book is closer to me, and we are on the opposite sides. I ask what happens with the pen as I put it behind the book. “ It disappears!” says Arina. “ The object that is closer can block the distant object “ – guesses Jordan. “ So, what about the clouds and the Sun? Do the clouds hide the Sun? – yes! – Does the Sun ever blocks the clouds? – The class says “ no”
- *It is very interesting that the kids of this age do not have this inner feeling of something in front of something being closer to us, though it is one of the common ways our brain uses our vision to communicate the distances.*
- Should we talk more about that with them, or should we wait for a couple of years? (This fact is obvious for 9 year olds)

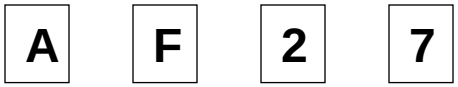
PROOF DISCUSSION -2

- Do not introduce real proofs – introduce fake ones!
- Theorem. A cat has nine tails.
- Proof. No cat has eight tails. A cat has one more tail than no cat. Therefore a cat has nine tails.

NEGATIVE CONFIRMATION

- It is known, that people tend to test in a one-sided way, by searching for evidence **consistent with their hypothesis**. They look for the evidence that they would expect to see if their hypothesis was true, **neglecting what would happen if it were false**.
- I was told, it is an evolutionary trait – if you think you see a tiger, you run away; if would not hurt you to run away, even if you are mistaken.

NEGATIVE CONFIRMATION



- Identify which cards you need to turn over in order to test the hypothesis:" If there is a vowel on one side of the card, there is an even number on another side of the card"

- Most of the people suggest turning A, which is an obvious choice confirming the hypothesis. However, none of my students would suggest turning 7, which **tests whether the hypothesis is false**. It is important to discuss this phenomena with the students, teaching them about their **inner preferences**, and advising them to re-assess their assumptions every time they approach a problem, **even if it looks counterintuitive.**

SOLVING BACKWARDS, AND AN IDEAL FINAL RESULT

- Solving backwards, and thinking of an ideal final result, are both powerful problem solving techniques. However, they seem to be hard for the kids.
- The best way of introduction for me was using the mazes, and solving them from finish to start.
- Another good source of such problems are Nim-type games.

GUESSING AND ESTIMATION

- Children love guessing! I use this a lot when we are studying probability distributions. Children see making predictions as a wonderful game. We also do a lot of estimations, thinking of **extreme cases**, and suggesting various **tests** to check out our predictions.

Example

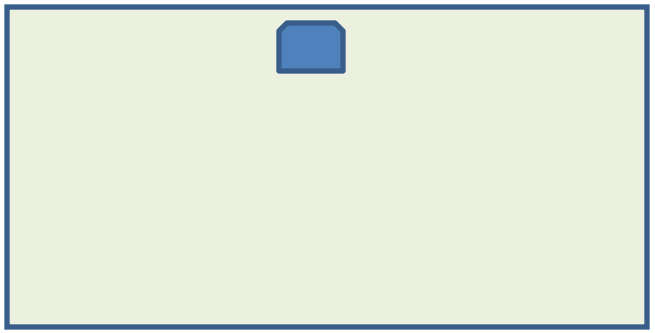
- I have 3 poker chips - blue, red and white inside the lunch bag. After 9 trials, am I going to have more reds, more blues or more whites? (write your guess on your paper)
- *Observations:*
- *It is interesting to notice that while older kids tend to go to a symmetrical 3-3-3 distribution (that never happened to us in class), the younger ones tend to favor one color over the rest, giving something like 6-2-1 distribution. Well, the truth, for the given number of trials, lies in between* ☺
- *Over several rounds, we never got the exact guess of any of the kids, to their big surprise - you may suggest them to count the number of all possible outcomes, and then calculate the possibility of their guess.*

SYMMETRY

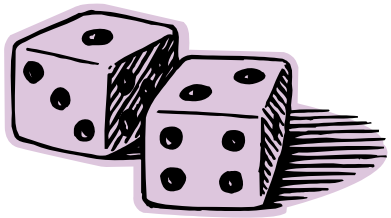
- ***"The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful." Aristotle***
- Symmetry has a **special appeal** to us as humans. We tend to notice it and to adore it. Why are we inclined to pay special attention to it? I am not a psychologist, and I am not well versed in this matter; however, I certainly observe this phenomenon in class.
- Children like to play with symmetry, to build symmetrical ornaments and to explore mirror and rotational symmetry. However, our admiration of symmetry sometimes leaves us to consider only symmetrical solutions. **Non-symmetrical ones come to us with much more difficulty.**

Example

- It is very easy to put **8 chairs** along the walls of a rectangular room such as there is an equal amount of chairs at each wall. No one will have any difficulty solving this problem. Now, suggest the kids to solve this problem for **10 chairs** instead.



GAME THEORY



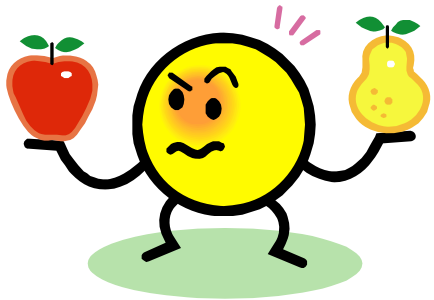
- **" The Ethical Equations, of course, link conduct with probability, and give mathematical proof that certain patterns of conduct increase the probability of certain kinds of coincidences. But nobody ever expected them to have any really practical effect" Murray Leistner, " The Ethical Equations"**
- I remember reading "The Ethical equations" as teenager, and I remember how fascinated I was with the idea that ethics and mathematics can relate. I could not resist giving a taste of this to the kids.
- I consider it critical to show the children that they are encouraged to **apply their knowledge in one domain to a very different one**; in this case, math to social studies.
- I also wanted to discuss the notion of **unintended outcomes** with them. It is extremely interesting and intriguing to the children to learn that <u>their desires may bring them to a very unexpected outcome.</u>

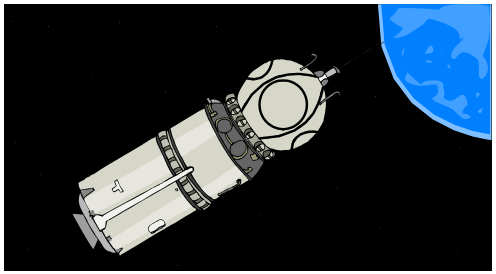
PRISONER'S DILEMMA –THE STORY

- Two kids (ask for volunteers, write the outcomes on the board) are in big trouble, but the teachers and parents do not know who is the mastermind behind the misbehavior.
- They separate the kids, and tell them: if we get no answer from both of you, you both get suspended for a day. If you tell that it was another kid, you are released and another kid gets 3 days of suspension. If both of you tell of each other, you both get 2 days.
- Tasya : it was Constantin! Constantin: It was Tasya!
- (discussion)

PRISONER'S DILEMMA –THE GAME

- Each pair is given red and blue poker chips. Players sit back to back, so they do not see each other's move. The players are NOT allowed to communicate! They simultaneously put the chip on the piece of paper on their side.
- If both play a red chip, they get 2 toy coins each. If they play different colors, the one who plays the blue gets 3 coins, and the one who plays the red gets nothing. If both play blue, both get 1 coin each. The purpose is to get as many coins as possible.
- Write your outcomes in the table (Besides everything else, this problem is good as a data collection and analysis practice.)
- Count your total score in coins. What strategy (cooperation or non-cooperation) is better in the short time? In the long time? (discussion)





SUMMARY

- As you see, I defined the minimal “success of the mission” as introducing the basic problem solving skills to the kids.
- I told you about my efforts to teach the children not only how to solve the problems, but also to show them that they need to learn **about themselves** in order to do it effectively.
- Good teachers do this intuitively, but it is a pity that this **self-awareness and problem solving skills** are not taught to the kids on a regular basis.

TEXTBOOK FOR MATH CIRCLE ORGANIZERS

- I decided to write a textbook for future organizers of the math circles for the elementary school kids. I plan to devote a chapter for problem solving skills, and the rest of the book to lesson plans for the math circle, along with sample responses, hints and references for problem solving techniques .

Content

ELEMENTARY MATH CIRCLE IN ACTION

- Watch the video at:
- http://www.youtube.com/watch?v=l3Jw3Evu9l0