Polynomials and Complex Numbers

Dimitar Grantcharov, Mid-Cities Math Circle

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1 Warming up problems

In the following two problems one may use the fact that if \( z \) is a root of a polynomial \( P(z) \) and \( P(z) = P(1/z) \), then \( 1/z \) is also a root of \( P(z) \).

Problem 1. Solve the equation \( z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0 \).

Problem 2. Solve the equation
\[
4z^{11} + 4z^{10} - 21z^9 - 21z^8 + 17z^7 + 17z^6 + 17z^5 + 17z^4 - 21z^3 - 21z^2 + 4z + 4 = 0.
\]

Problem 3. (a) Find a polynomial with integer coefficients whose zeros include \( \sqrt{2} + \sqrt[3]{5} \).

(b) Find a polynomial with integer coefficients whose zeros include \( \sqrt{2} + \sqrt[3]{5} + \sqrt[5]{7} \).

Remark. Obviously Problem 3 (b) is substantially more difficult than Problem 3 (a). What would be the minimal degree of a polynomial one of the roots of which is \( \sqrt{2} + \sqrt[3]{5} + \sqrt[5]{7} \)?

Problem 4. Determine \( a, b \), so that \( (x - 1)^2 \) divides \( ax^4 + bx^3 + 1 \).

2 Division with quotient and remainder

Division of polynomials. For any polynomials \( f(x) \) and \( g(x) \) there exist unique polynomials \( q(x) \) and \( r(x) \) such that
\[
f(x) = g(x)q(x) + r(x), \quad \deg r < \deg g \text{ or } r(x) = 0.
\]
For example, if \( f(x) = x^7 - 1 \) and \( g(x) = x^3 + x + 1 \) then the quotient \( q(x) \) is \( x^4 - x^2 - x + 1 \) and the remainder \( r(x) \) is \( 2x^2 - 2 \). In the case \( g(x) = x - a \) we obtain an important fact: \( f(a) = 0 \) if and only if \( f(x) = (x - a)q(x) \) for some polynomial \( q(x) \).

The coefficients of the polynomials can be in \( \mathbb{C}, \mathbb{R}, \mathbb{Q}, \) or \( \mathbb{Z} \). In the case of \( \mathbb{Z} \) we may have a situation when the quotient and the remainder are not with integer coefficients. Take for example \( f(x) = x^2 \) and \( g(x) = 2x + 1 \). Is there any such problem with \( \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \)?

**Problem 5.** Find the remainder of \( x^{81} + x^{49} + x^{25} + x^9 + x \) when divided by \( x^3 - x \).

**Problem 6.** Find the remainder of \( x^{1959} \) when divided by \( (x^2 + 1)(x^2 + x + 1) \).

**Problem 7.** Let \( f(x) = x^4 + x^3 + x^2 + x + 1 \). Find the remainder of \( f(x^5) \) when divided by \( f(x) \).

**Problem 8.** Let \( p(x) \) be a polynomial with integer coefficients. Assume that \( p(a) = p(b) = p(c) = -1 \), where \( a, b, c \) are three different integers. Prove that \( p(x) \) has no integer zeros.

**Problem 9.** Let \( P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) be a polynomial with integer coefficients. Suppose that there exist four distinct integers \( a, b, c, d \) with \( P(a) = P(b) = P(c) = P(d) = 5 \). Prove that there is no integer \( k \) with \( P(k) = 8 \).

**Problem 10.** (USAMO 1975) If \( P(x) \) denotes a polynomial of degree \( n \) such that \( P(k) = k/(k+1) \) for \( k = 0, 1, 2, \ldots, n \), determine \( P(n+1) \).

### 3 Polynomial equations

**Problem 11.** Find all polynomials \( P(x) \) for which \( xP(x-1) = (x+1)P(x) \).

**Problem 12.** Determine all polynomials \( P(x) \) such that \( P(0) = 0 \) and \( P(x^2 + 1) = P(x)^2 + 1 \).

**Problem 13.** Find all polynomials \( P(x) \), for which \( P(x)P(2x^3) = P(2x^3 + x) \).
Problem 14. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a function such that \( f(z)f(iz) = z^2 \) for all complex numbers \( z \). Prove that \( f(z) + f(-z) = 0 \) for all complex numbers \( z \).

Problem 15. Find all polynomials \( P(x) \), for which \( P(x)P(2x^2) = P(2x^3 + x) \).

Problem 16. Find all polynomials \( P(x) \), for which \( P(x^2) + P(x)P(x+1) = 0 \).

4 Irreducibility of polynomials

Problem 17. Factor the following polynomials as products of irreducible polynomials with integer coefficients.

(a) \( x^4 + x^2 + 1 \), (b) \( x^{10} + x^5 + 1 \), (c) \( x^9 + x^4 - x - 1 \).

Problem 18. Prove that \( (1 + x + \ldots + x^n)^2 - x^n \) is the product of two polynomials.

Problem 19. If \( a_1, \ldots, a_n \) are distinct integers, prove that \( (x-a_1)\ldots(x-a_n) - 1 \) is irreducible.