Investigating the Mathematics of Sona: Sand Drawings from Angola
Instructor Notes

This lesson has an easy point-of-entry and should be well-suited to early meetings of a newly formed group. Most of the information presented, and all of the graphics, are from [1].

Goals of the lesson include investigation, forming conjectures, and precisely communicating mathematical ideas. Problem-solving strategies instructors can anticipate emphasizing include:

- Consider simple examples
- Exploit symmetry
- Solve a simpler problem

Mathematical ideas that readily appear include:

- Greatest Common Divisor
- Graph Theory: what is a graph, Euler's Formula
- Sums of arithmetic sequences

Note that the array of points for *sona* may be rectangular or some other shape. It may be easiest to focus on rectangular arrays to start.

If you want to ask participants to write an algorithm for drawing rectangular *sona*, it will be best to show the execution of a few drawings and encourage participants to observe carefully.

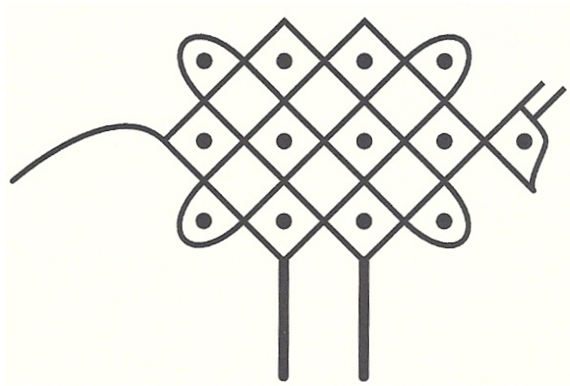
For the Cokwe, these drawings are their writings. It is a particular writing, without letters, without an alphabet. It is a language composed of points and lines....After cleaning and smoothing the ground, the story teller, who is at the same time the drawer, first marks points with the tips of his extended fingers. He uses the index and ring fingers of his right hand.

To mark points from right to left, he keeps the tip of his ring finger on the point last marked on the ground, while marking a new point with the index finger guaranteeing that the distance between two consecutive points of a row is always the same.

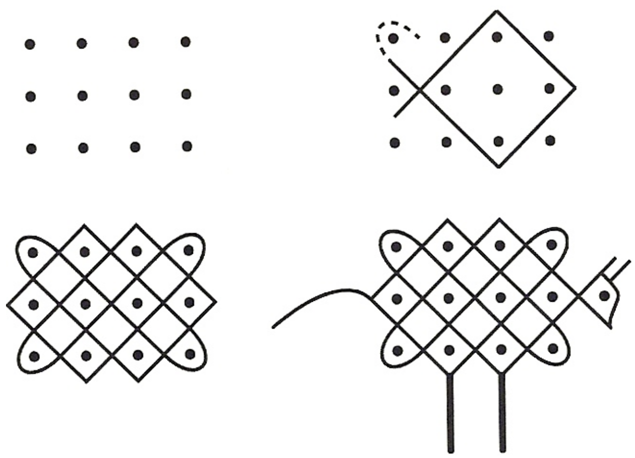
In this way, the Cokwe mark in the same a rectangular array of little points. The rows are perpendicular to the columns of points. (Depending on the drawing to be executed, sometimes it is necessary to mark additional points in the centres of the squares of the array points.)

Once all necessary points are marked, one begins the execution of the figure. The Cokwe are accustomed to drawing in the sand with the index finger of the right hand. [1]

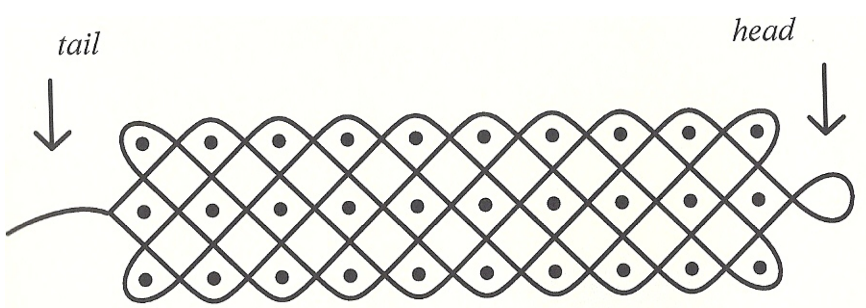
The following figure is an antelope. Note that the antelope requires no "additional points."



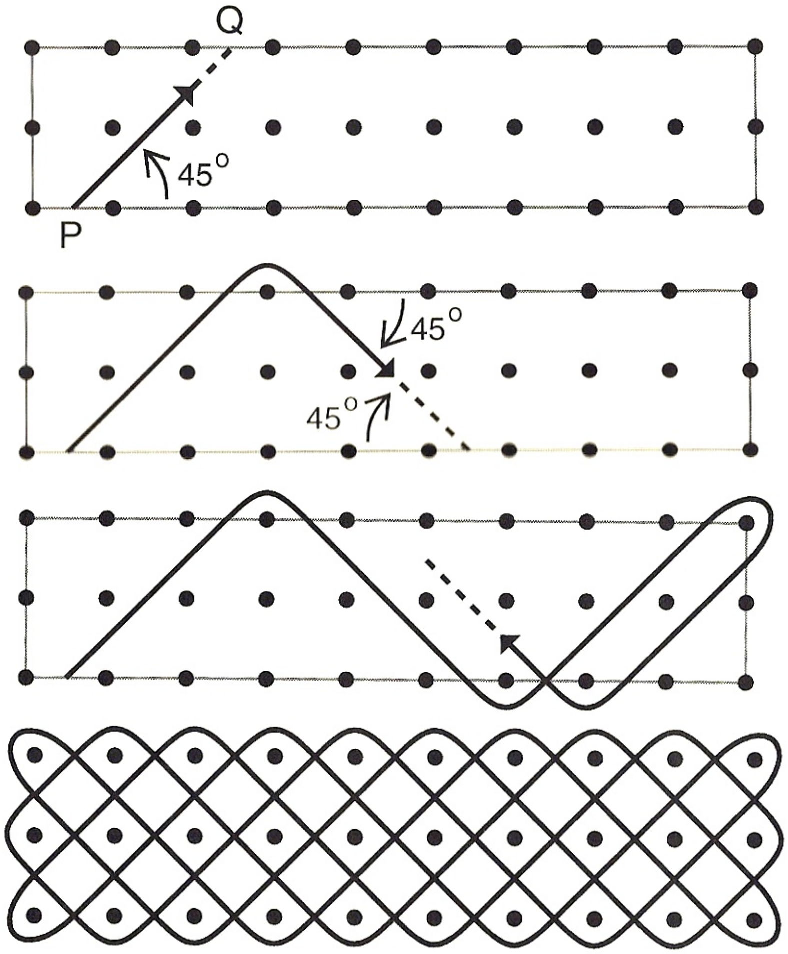
The storyteller does not retrace portions of lines or lift his finger from the sand unless absolutely necessary; the storyteller may draw a curve that intersects itself. Note that external features such as the head, tail, and legs are drawn last. The following graphic illustrates the procedure:



Another figure built from a rectangular array of points is the lioness:

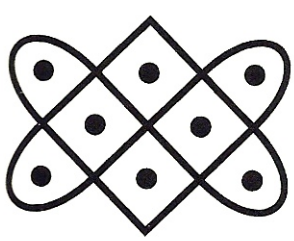


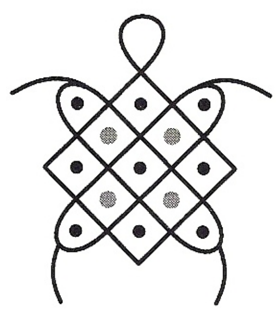
The following graphics show the procedure for drawing the lioness:



The head and tail would then be added last.

Two examples of *sona* with additional points are *vusamba* (friendship) and the tortoise:





Notice that three closed curves are required for the tortoise, whereas all other *sona* we have seen have been drawn with a single closed curve.

Good questions to ask participants. (Of course, it would be best to ask them to generate a list of questions, see question 3):

1. Select one of the *lusona* drawings. Write an algorithm (sequence of steps or instructions) telling how to draw the figure.

2. Examine the *sona* drawings we have seen so far. What are their similarities and differences?

3. Try to make some of your own *sona* drawings, being attentive and making observations as you do so. What questions present themselves?

4. What do you notice if you focus on rectangular arrays of points?

5. What happens if you augment a *lusona* built on a rectangular array you already understand with a square array of points on one side? (Theorem 2 in [3] tells us that if the original *sona* used one closed curve, the new arrangement will also.)

6. How will the drawing of the lioness change if the number of points in the original array changes?

7. How is the tortoise different from the other *sona* we have seen?

8. Do you notice anything about the diagonals of the rectangular *sona* we have seen?

   Be prepared for questions such as the following:

   - How many turns are required to complete a *lusona* on a rectangular array with n points?
   - How many straight line segments are required to complete a *lusona* with on a rectangular array n points?
   - How many intersections are required to complete a *lusona* on a rectangular array with n points?
   - What is the effect of adding additional points?
   - What kinds of symmetry do these drawings have?
   - Does it matter where you start drawing a curve?
   - How many drawings on a given set of points are possible? (This is an open question for general n)
   - How many bounded regions are made?
   - How many closed curves are required?
   - When can a single curve be used?
   - Can a *lusona* with a single closed curve be used for any number of points?

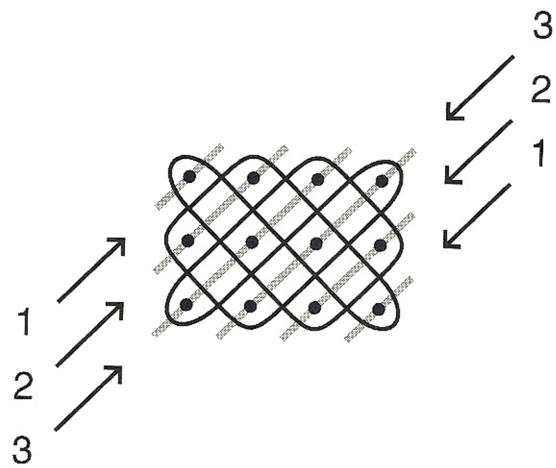
The last three questions may prove to be the most straightforward. The final question is easily solved, since positioning the dots in a straight line gives a drawing enclosing all points using a single closed curve. The questions of "How many closed curves are required?" and "When can a single closed curve be used?" are answered for rectangular arrays of m by n dots by noticing that the number of closed curves required is $\gcd(m,n)$. So, a single closed curve can be used when the dimensions are relatively prime (see Theorem 1 in [3]).

The second and third questions are answered by Lemma 1 of [2], under certain circumstances (one point per bounded region, no points excluded from a bounded region). This is proved by translating a *sona* into a traditional mathematical graph (in the graph-theoretic sense) in the following way: each curve intersection maps to a vertex in the graph and each segment of a curve connecting two intersections (vertices) maps to an edge in the graph. Each bounded region becomes a face; the "exterior" of the *sona* is considered an infinite outside face. Once this graph-theoretic translation has been made, Euler's formula

$$V - E + F = 2$$

can be used. This graph-theoretic interpretation opens many avenues of further exploration.

With regard to the question about diagonals of the rectangular *sona*, Gerdes' book [1] includes the following exploration recalling the 3×4 array used in the antelope:



Rearranging the original array, we obtain two superimposed squares each of size 3. Note that the number of points used is

$$2(1 + 2 + 3) = 3 \times 4$$

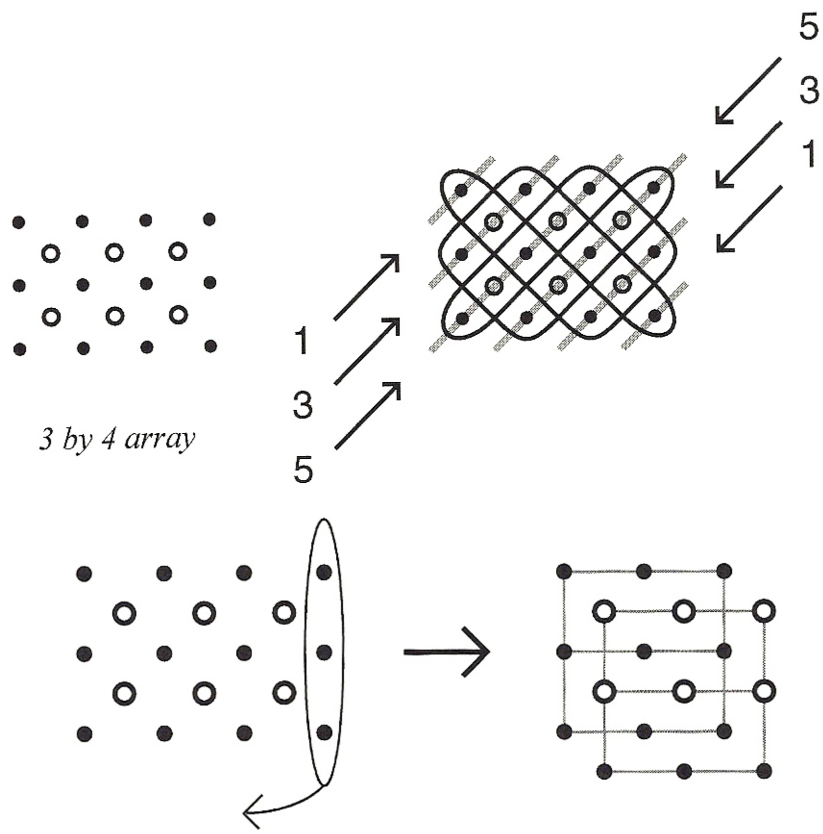
so that

$$1 + 2 + 3 = \frac{3 \times 4}{2}$$

This generalizes to any $n \times (n+1)$ array, because, for each n there are two diagonals containing n points. This gives another way to find the sum formula for an arithmetic sequence with first term 1 and common difference 1.

In addition, one can find the sum of an arithmetic sequence with first term 2 and common difference 2 (even numbers) by carrying out the multiplication by 2 on the left side of the first equation given above.

For an arithmetic sequence of first term 1 and common difference 2 (odd numbers), notice the following illustration using diagonals for rectangular arrays with additional points:



Notice that the number of points is

$$2(1+3+5) = 2 \times 3^2$$

so that the sum of the first 3 odd numbers is 3^2. This generalizes, using an $n \times (n+1)$ array with additional points, to give that the sum of the first n odd numbers is n^2.

References

[1] Paulus Gerdes, *Drawings from Angola: Living Mathematics*, Morrisville, NC: Lulu.com, 2007.

[2] Erik D. Demaine, Martin L. Demain, Perouz Taslakian, Godfried T. Toussaint, *Sand Drawings and Gaussian Graphs*, Journal of Mathematics and the Arts, Volume 1, Issue 2, June 2007, pp. 125–132.

[3] D. Chavey, *Sona Geometry*,
http://www.beloit.edu/computerscience/faculty/chavey/sona/

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