GCD, LCM, Division algorithm & The Euclidean Algorithm

OMSI Math Circle for Sixth through Ninth Graders
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The problems below are taken mostly from the following books. Several of them have been modified to suit the needs of this session.

Mathematical Circles (Russian Experience) by Fomin, Genkin & Itenberg
The Stanford Mathematics Problem Book by Polya, Kilpatrick
The Art and Craft of Problem Solving by Paul Zeitz

Warm Up Questions (to be described in class):

Definitions & Notation:

GCD, denoted by (a,b)**:** Given two natural numbers a,b , their greatest common divisor or GCD is defined to be the largest positive integer that divides both a and b

LCM, denoted by $[a,b]$**:** Given two natural numbers a,b , their least common multiple or LCM is defined to be the smallest positive integer that is a multiple of both a and b

Relatively Prime or Coprime: If the GCD of two natural numbers a,b equals 1, then the two numbers are said to be **relatively prime or coprime.**

We say b divides a [denoted by $b \mid a$] if there exists an integer c such that $a = bc$

Theorems:

1. Division Algorithm for positive integers: For to positive integers a,b, where $b \geq a$, there exist unique integers q,r with $q \geq 1$ and $0 \leq r < a$ such that $b = qa + r$

2. GCD of two natural numbers of a and b is the **smallest** positive linear combination of a and b.

3: If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$

4. Fundamental Theorem of Arithmetic (FTA): Every natural number greater than 1 can be **uniquely** expressed as a product of prime numbers in increasing order.

If you are familiar with all the problems in part A, then move on to parts B & C directly.

Part A: For the following problems, you may assume FTA if you like.

1. Decide if each statement about an integer N below is true or false. If false, give a counterexample.

(a) If $4 | N$ and $3 | N$, then $12 | N$

(b) If $4 | N$ and $6 | N$, then $24 | N$

2. If an integer N is not divisible by 3, then is it possible for $2N$ to be divisible by 3?

3. If $5N$ is divisible by 3, then is it true that N to be divisible by 3?

4. If $15N$ is divisible by 6, then is it true that N to be divisible by 6?

5. List all the different divisors of $5^3 \cdot 7^4$.

6. Suppose that p and q are different prime numbers. For each number below, find the total number of different divisors, (I) pq (II) p^2q (III) p^2q^2 (IV) p^mq^n

7. Compute the following in the most efficient way you can think of and present you answer in the manner you think makes most sense for the given problem..

(I) $GCD(2^8 3^{14} 7^5 13, 3^5 7^7 13^2 19^3)$

(II) $GCD(1381955, 690713)$

(III) $GCD(2n+13, n+7)$

(IV) $GCD(2^{100}-1, 2^{120}-1)$

(IV) $GCD\left(\underbrace{111...111}_{100 \text{ of them}}, \underbrace{11...111}_{60 \text{ of them}}\right)$

Part B: For the following problems, you may assume FTA if you like.

1. Can a number that is written using exactly one hundred each of "0" and "1" and "2" be a perfect square. Explain your answer.

2. Prove (formally) that there are infinitely many primes.

3. Show that $(a,b)[a,b] = ab$ for any positive integers a,b

3. Show that if $d | a$ and $d | b$, then $d | ax+by$ for any integers x, y

4. Consecutive integers are always relatively prime.

5. If there exist integers x, y such that $ax+by=1$ then a and b are relatively prime.

6. Explain why the Euclidean algorithm works! How might you program it?

Part C:

1. Show that $1+\frac{1}{2}+\frac{1}{3}+....\frac{1}{n}$ can never be an integer.

2. Prove FTA. You may assume the theorems listed before FTA on the first page.

3. (AIME 1985) The numbers in the sequence 101, 104, 109, 116, …… are of the form $a_n = 100+n^2$, where $n = 1,2,3,......$ For each n, let $d_n = (a_n, a_{n+1})$. Find the maximum value of d_n as n ranges through the positive integers.

4. Prove that the only solution of the equation $x^2+y^2+z^2 = 2xyz$ in integers x, y and z is the trivial solution $x = y = z = 0$

Web sites to explore:

Euclidean algorithm: http://en.wikipedia.org/wiki/Euclidean_algorithm

Euclid's book "Elements" online:http://aleph0.clarku.edu/~djoyce/java/elements/toc.html