The Teachers' Circle **Mar. 20, 2012**

HOW TO GAMBLE IF YOU MUST

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(I'll bet you \$5 that if you give me \$10, I'll give you \$20.)

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Basic Laws and Definitions of Probability

1 If A is an event, then the probability $P(A)$ of the event occurring is a number that is between 0 and 1. If $P(A) = 0$, the event cannot occur; if $P(A) = 1$, then the event is certain.

2 The total of the probabilities of all *disjoint* events is 1. (Disjoint means mutually exclusive.)

3 *Addition Rule*: If A and B are disjoint events, then $P(A \text{ xor } B) = P(A) + P(B)$. Note that we are, redundantly, specifying "xor" instead of the more ambiguous "or."

4 *Complement Rule*: $P(\text{not } A) = 1 - P(A)$.

5 *Conditional Probability*: Let $P(A|B)$ denote the probability of A, *given* B. Then

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$

6 *Multiplication Rules*:

(a) The conditional probability formula above gives us a formula for $P(A \text{ and } B)$ that is *always true*:

$$P(A \text{ and } B) = P(A|B)P(B).$$

(b) In particular, as long as A and B are *independent* events, then we have a simpler formula:

$$P(A \text{ and } B) = P(A) \times P(B).$$

Many errors in probability come from people using (b) when they should use (a)!

7 *Expectation*. A **random variable** is a function ("machine") which outputs numbers, where each output value has a probability. For example, the number you see on the top of a die when you toss it is a random variable, where each of the numbers $1,2,3,4,5,6$ has a probability of $1/6$. The sum of two dice is a different random variable, with 11 outputs: $2,3,4\ldots,12$, with probabilities ranging from $1/36$ (for 2 or 12) up to $1/6$ (for an output of 7).

If X is a random variable, we define the **expected value** or *expectation* of X, denoted by $E(X)$, to be the "average" output of X. Imagine that X is a machine that spits out values and you just let it run for many values, and then take the average. That should approximate $E(X)$. The theoretical value for $E(X)$ is just the sum of the products of each value multiplied by the probability that it attains that value.

For example, suppose X outputs 5 and -2 with probabilities 0.3 and 0.7, respectively. Then $E(X) = 5 \cdot 0.3 + (-2) \cdot (0.7) = 0.1$.

Fun with Dice

1 *Thirty-six scenarios*. When two dice are rolled, there are 36 different outcomes, because $36 = 6 \times 6$. Each outcome is equally likely. This allows us to compute probabilities easily. Use this table to compute the various sums that can occur. I filled in the first few cells.

	1	2	3	4	5	6
1	2	3	4			
2		4				
3						
4						
5						
6						12

(a) Verify that the probability of rolling two dice and getting a sum of 2 is $1/36$.

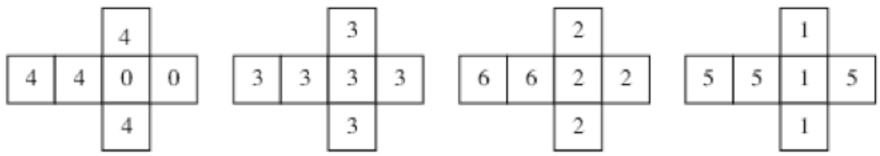
(b) What is the probability of rolling two dice and getting a sum of 5?

(c) What is the most likely sum, and what is its probability?

(d) Test these probabilities by experiment. For example, suppose you want to test for a sum of 12, which should also have a probability of $1/36$. You'd expect to see this sum happen only once every 36 or so rolls. Experiment!

2 *Non-transitive dice*. Construct the following set of four dice.

A B C D



(a) Suppose one person tosses the A die, and the other tosses the B die. The winner is the one whose die has the bigger number. What is the probability that A wins? You will need to make a 6×6 table like this. I filled in a few cells, saying who the winner is.

A vs B	0	0	4	4	4	4
3	B	B	A			
3						
3						
3						
3						
3						

(b) Work out the probabilities for the other possibilities, such as B vs. C, C vs. D, etc. Here are some tables to help you organize your work.

B vs. C	2	2	2	2	6	6
3	B	B			C	
3						
3						
3						
3						
3						

C vs. D	2	2	2	2	6	6
1	C	C				
1						
1						
5						
5	D					
5						

A vs. C	2	2	2	2	6	6
0						
0						
4						
4						
4						
4						

A vs. D	1	1	1	5	5	5
0						
0						
4						
4						
4						
4						

(c) Now that you have collected data, do you notice something strange about these four dice? Can you construct a "sucker bet" with them?

3 What's the probability of rolling six dice and

(a) getting a sum of 6?

(b) getting a sum of 7?

(c) getting a sum of 10?

(d) having all six numbers be equal?

(e) having all six numbers be different?

4 (a) On average, how many times must a die be thrown until one gets a 6?

(b) How many times, on average, should one toss a fair die in order to see all 6 possible outcomes?

Hint: Conditional Probability

5 Suppose a test for HIV has a 99% accuracy probability; i.e., there is a 1% chance that the test gives the wrong answer. Suppose that in a certain population, the probability of actually having HIV is 0.5%. Now suppose you are a member of this population, and you take the HIV test, and you get a "positive" result. What is the probability that you actually have HIV?

6 Two cards are placed in a hat. One card is red on one side, and black on the other side. The other card is black on both sides. The hat is shaken, and someone draws a card, showing you one side. It is black. What is the probability that the other side is black?

Fun with Cards

7 *A Feel for Cards*. I deal cards onto a table, while you tell me whether the card should be face up or face down (I do what you say). You also tell me when to stop dealing the cards (it doesn't have to be the whole deck). Then you put a blindfold on me. I bet you that my fingers are so sensitive that I can detect the difference between a face up card and a face down card by touch. I bet that I can divide the cards on the table into two piles, each of which has exactly the same number of face cards. Do you want to take this bet?

8 *Face Cards*. A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a "face card;" i.e., a Jack, Queen, or King? Do you want to take this bet?

9 *The First Ace*. Shuffle a deck of cards, and deal out the cards until you see the first ace. Define the random variable F to be 1 if the first ace is the 1st card, 2 if it is the 2nd, etc. Hence F will output numbers in the range 1 to 49, inclusive.

(a) What it the most likely output value?

(b) What is the average output value? (In other words, what is $E(F)$?)

10 *Kruskal Count*. This one is hard to describe. I will demonstrate this amazing trick for you. Your job will be to explain why it works.

Expectation and Random Walks

11 *Two Lottery Tickets*. It costs a consumer \$1 to buy a Klopstockia lottery ticket. The buyer then scratches the ticket to see the prize. Compute, to the nearest penny, the expected profit that the state of Klopstockia makes per ticket sold, given the following scenarios for prizes awarded. (The state will make a profit if the expected value of the lottery ticket is *less* than \$1.)

(a)

Prize	\$1	\$10	\$1000
Probability	$\frac{1}{10}$	$\frac{1}{1,000}$	$\frac{1}{1,000,000}$

(b)

Prize	\$1	\$10	\$1000	a free lottery ticket
Probability	$\frac{1}{10}$	$\frac{1}{1,000}$	$\frac{1}{1,000,000}$	$\frac{1}{5}$

12 Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

(a) the game never ends?

(b) the first player wins?

(c) the second player wins?

13 *Hint: Expectation is Additive*.

(a) Place n letters at random into n envelopes. What is the average number of letters which get into the correct envelopes?

(b) An urn contains 1000 balls, labeled $1, 2, 3, \ldots, 1000$. You perform the following procedure: mix the balls well, pick a ball, record its number, and then put it back in the urn. If you do this procedure 1000 times, what is the average number of *distinct* integers that you will record?

14 *The Classic Gambler's Ruin Problem*. Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars, and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e., has zero dollars) the game is over. What is the probability that A goes bankrupt?

What happens if the probabilities are not equal; i.e., what if the probability that the coin is heads is p, for some fixed $0 \leq p \leq 1$.

15 *What a Loser!* You arrive in Las Vegas with \$100 and decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money. Which of the following strategies is best?

(a) Making bets of \$1 each time.

(b) Making bets of \$10 each time.

(c) Making a single bet of \$100.

16 *A Gambling "System."* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$ depending on whether you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. You are guaranteed to make a profit? Right? Use expectation to show that this won't work. What if you triple instead of double?

17 *The St. Petersburg Paradox*. Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I'll pay you \$2. If it takes two flips, then I'll pay you \$4. Three flips, \$8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

How much is this game worth *to you*? In other words, if a there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you'd pay at least 1 dollar. In fact, you'd almost certainly pay 2 dollars. How about 3? 4? 5? More?