Crack the Graham

In this activity you will try to crack some of the secrets of a sequence discovered by Ron Graham. Consider the following conditions:

A. $a_n(6) = a_a, a_2, \cdots$ is a sequence of positive integers.

B. $a_n(6)$ is a finite sequence.

C. $a_n(6)$ starts with 6.

D. $a_n(6)$ is increasing.

E. The product of the terms of $a_n(6)$ is a perfect square.

1. Given five arbitrary conditions (A, B, C, D, and E) how many different possibilities are there for which ones are true? e.g. A, C, and D could be true, or they could all be true, or none of them could be true.
2. Write out at least five combinations of the properties A - E. Find five sequences so that each one satisfies exactly the properties (and no other among A - E) in one of your five combinations.
3. Find four sequences that satisfy all five conditions A - E.

The above properties can be generalized for sequences starting with any positive integer m. In this case we will call the properties A(m), ... E(m). Using these properties, Ron Graham defined the following funny looking sequence:

$$g_m = \min(\max(\{a_n(m) \,|\, \mathrm{A}(m), \ldots \mathrm{E}(m)\})\,.$$

4. Find the maximum of each of your sequences from question 3.
5. Find the minimum of all possible maximum values.
6. How can you know that you found the minimum of all possible maximum values, and did not over look some other sequence satisfying A(6) - E(6) with a smaller largest number?
7. Compute the first ten terms of the Graham sequence?
8. Does 20 appear in the Graham sequence somewhere? If so, where? If not, why not?
9. Does 7 appear in the Graham sequence somewhere? If so, where? If not, why not?
10. Does 120 appear in the Graham sequence somewhere? If so, where? If not, why not?
11. Can you decide exactly which numbers appear in the Graham sequence? Think about it. Make guesses. Then try to make a case that your best guess is correct that would win in a court of law.