DOTS and DASHES

This is a sample (draft) chapter from:

MATHEMATICAL OUTPOURINGS

Newsletters and Musings from the St. Mark's Institute of Mathematics

James Tanton
www.jamestanton.com

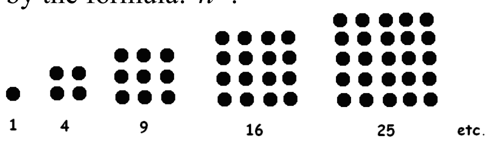
This material was – and can still be – used as the basis of a successful MATH CIRCLE activity.

The St. Mark's
Institute of Mathematics
Newsletter

JANUARY 2007

THIS MONTH'S PUZZLER:

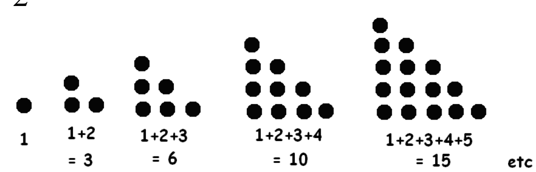
1. The sequence of square numbers begins 1, 4, 9, 16, 25, … and the n-th square number is given by the formula: n^2.



The sequence of non-square numbers begins 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, ….

a) What's the 100-th non-square number?
b) Find a formula for the n-th non-square number

2. The sequence of triangular numbers begins 1, 3, 6, 10, 15, … and the n-th triangle number is $\frac{1}{2}n(n+1)$.



The sequence of non-triangular numbers begins: 2, 4, 5, 7, 8, 9, 11, …

a) What's the 100-th non-triangular number?
b) Find a formula for the n-th non-triangular number.

TODAY'S TIDBIT:

Consider the list of prime numbers:

$\{p_n\}$: 2, 3, 5, 7, 11, 13, 17, …

(Here p_n means the n n-th prime).

Let q_n count the number of primes less than n. For example, $q_{10} = 4$ because there are 4 primes less than ten. This gives the "frequency sequence" for the primes:

$\{q_n\}$: 0, 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, …

Now let's compute the frequency sequence of the frequency seqeunce! There are 2 numbers among $\{q_n\}$ less than one, 3 numbers less than two, 5 numbers less than three, 7 numbers less than four, 11 numbers less than five, and so on. If you keep checking you see that the original sequence of primes reappears!

$\{p_n\}$: 2, 3, 5, 7, 11, 13, 17, …

More is true! Take the sequence $\{p_n\}$ and its frequency sequence $\{q_n\}$ and add 1 to the first element of each, 2 to the second element of each, 3 to the third, 4 to the fourth, and so on. We obtain the sequences:

$\{P_n\}$: 3, 5, 8, 11, 16, 19, 24, 27, 32, …
$\{Q_n\}$: 1, 2, 4, 6, 7, 9, 10, 12, 13, 14, …

[So here $P_n = p_n + n$ and $Q_n = q_n + n$.]

Notice that *ALL* the counting numbers 1, 2, 3, 4, … appear split among these two sequences with no repeats! Mathematicians call these "complementary sequences."

The lovely phenomenon is not unique to the list of prime numbers.

Theorem: *Write down any sequence of non-negative whole numbers that never decreases. (You may repeat entries multiple times.) Call the sequence* $\{p_n\}$.

i) The frequency sequence of the frequency sequence of $\{p_n\}$ *is* $\{p_n\}$.

ii) Adding position numbers to the entries of $\{p_n\}$ and to the entries of its frequency sequence always produces complementary sequences.

Let's try this on the following sequence (which I just made up!):

$\{p_n\}$: 1, 2, 2, 2, 3, 3, 6, 7, 7, 7, 7, 9, 11, 11, 14, 14, 14, 14, 14, 14, 14, 14, 15, ….

Its frequency sequence is:

$\{q_n\}$: 0, 1, 4, 6, 6, 6, 7, 11, 11, 12, 12, 14, 14, 14, 22, …

(There are 0 entries in $\{p_n\}$ less than one, 1 entry less than two, 4 entries less than three, and so on.)

Now compute the frequency sequence for $\{q_n\}$ to see that is $\{p_n\}$. (Do this!)

Let's add the position numbers 1, 2, 3, 4, 5, … to each of the sequences $\{p_n\}$ and $\{q_n\}$. This gives:

$\{P_n\}$: 2, 4, 5, 6, 8, 9, 13, 15, 16,17, 18, 21, 24, 25, 29, 30, 31, 32, 33, 34, 35, 36, 38, …

$\{Q_n\}$: 1, 3, 7, 10, 11, 12, 14, 19, 20, 22, 23, 26, 27, 28, 37, …

Yep. Complementary!

EXPLANATION: In 2005, three St. Mark's students, Charles Zodda, Eric Rudyak, and Jae Shin You, came up with the following innovative proof:

Any sequence $\{p_n\}$ can be encoded as a string of dots and dashes. For example, the sequence 1, 2, 2, 2, 3, 3, 6, 7, 7, 7, 7, 9, 11, 11, 14, 14, 14, 14, 14, 14, 14, 14, 15, …. is encoded as:

* | * | | | * | | * * * | * | | | | * * | * * | | * * * | | | | | | | * | …

Here the first dash has 1 dot to its left, the second dash has 2 dots to its left, the third dash has 2 dots to its left, the fourth 2 dots, the fifth 3 dots, the sixth 7 dots, and so on. (Thus the *n*th dash is placed so that it has p_n dots to its left.) In this diagram:

p_n = number of dots to the left of the n th dash.

An entry q_n of the frequency sequence is given as the number of entries of $\{p_n\}$ with value less than *n*. With regard to the diagram, this is the number of dashes with less than *n* dots to their left. If you think about it, this means:

q_n = the number of dashes to the left of the n th dot

This statement is precisely the definition of p_n with the words "dot" and "dash" interchanged. Thus "frequency" means "interchange symbols." So the dot-dash diagram for the frequency sequence must be the original diagram with dots and dashes switched. (Try this! Draw the dot-dash diagram for 0, 1, 4, 6, 6, 6, 7, 11, 11, …) Doing this twice (taking the frequency of the frequency) clearly returns us to the original diagram and hence to the original sequence!

Further: Notice that the n th dash has $n-1$ dashes to its left and, by definition, p_n dots to its left. Thus the *n*th dash is in position $(n-1)+p_n+1 = p_n+n$. Interchanging the words "dots" and "dashes" tells us, by the same reasoning, that the n th dot lies in position q_n+n. Thus, if we list all the counting numbers along the top of a dot-dash diagram, the locations of all the dashes give us the values of the sequence given by $P_n = p_n+n$ and the locations of the stars the values of the sequence given by $Q_n = q_n+n$. The list of counting numbers is indeed split among two sequences given by the positions of the dots and the dashes!

RESEARCH: Can anything of interest be deduced from diagrams composed of strings of three symbols? How about two-dimensional arrays of symbols?

St. Mark's Institute of Mathematics.
mathinstitute@stmarksschool.org

DOTS and DASHES
January 2007

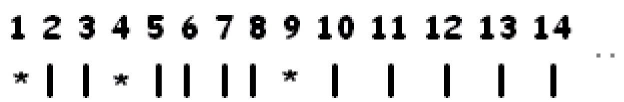
COMMENTARY, SOLUTIONS and THOUGHTS

The dots and dashes method allows us to find formulas for the non-square and non-triangular numbers.

Let $\{P_n\}$ be the sequence of square numbers and $\{Q_n\}$ be its complementary sequence, the non-squares:

$\{P_n\}$: 1, 4, 9, 16, …
$\{Q_n\}$: 2, 3, 5, 6, 7, 8, 10, …

We can think of these sequences as arising from a diagram of dots and dashes:



For each n, set $p_n = P_n - n = n^2 - n$ and $q_n = Q_n - n$. Given the work in the newsletter we have just arranged matters so that $\{q_n\}$ is the frequency sequence of $\{p_n\}$.

q_n = the number of entries of $\{p_n\}$ with value less than n
= the number of values k so that $p_k < n$
= the number of values k so that $k^2 - k < n$

Since we are speaking only of integers, $k^2 - k < n$ also means that $k^2 - k + \frac{1}{4} < n$. Thus:

q_n = the number of values k so that $k^2 - k + \frac{1}{4} < n$

= the number of values k so that $\left(k - \frac{1}{2}\right)^2 < n$

= the number of values of k so that $k < \sqrt{n} + \frac{1}{2}$

$= \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor$

where $\lfloor x \rfloor$ means "round x down to the nearest integer." If we take $\langle x \rangle$ to mean "round x (up or down) to the nearest integer" then a little thought shows:

$$\left\lfloor x + \frac{1}{2} \right\rfloor = \langle x \rangle$$

Thus we have a formula for q_n:

$$q_n = \left\langle \sqrt{n} \right\rangle$$

and hence also for Q_n, the nth non-square number:

$$Q_n \;=\; q_n + n \;=\; \left\langle \sqrt{n} \right\rangle + n \;=\; \left\langle \sqrt{n} + n \right\rangle$$

For example, the 100^{th} non-square number is $\langle 10 + 100 \rangle = 110$.

CHALLENGE: Derive this formula for the nth non-square number by an alternative method, one that doesn't invoke results about frequency sequence. (St. Mark's Institute followers came up with a number of different derivations.)

For the non-triangular numbers … Let $\{P_n\}$ be the sequence of triangular numbers and $\{Q_n\}$ its complementary sequence of non-triangles:

$\{P_n\}$: 1, 3, 6, 10, 15, …
$\{Q_n\}$: 2, 4, 5, 7, 8,…

We know $P_n = \frac{1}{2}n(n+1)$. We want a formula for Q_n.

Set $p_n = P_n - n = \frac{1}{2}n^2 - \frac{1}{2}n$ and $q_n = Q_n - n$ and again $\{q_n\}$ is the frequency sequence of $\{p_n\}$. Thus:

q_n = the number of values k so that $\frac{1}{2}k^2 - \frac{1}{2}k < n$
= the number of values k so that $k^2 - k < 2n$

The argument is now the same as the one above, except we are working with the value $2n$ instead of just n. This gives the nth non-triangular number as:

$$Q_n = \left\langle \sqrt{2n} + n \right\rangle$$

For example, the 100th non-triangle number is $\left\langle \sqrt{200} + 100 \right\rangle = 114$.

CHALLENGE: If $\left\langle \sqrt{n} + n \right\rangle$ gives the non-square numbers and $\left\langle \sqrt{2n} + n \right\rangle$ gives the non-triangular numbers, what non-numbers are given by $\left\langle \sqrt{3n} + n \right\rangle$?

CHALLENGE: What's the n th non-cube? What's the n th non-k th power (for $k \geq 4$)?

BEATTY SEQUENCES

In 1926 Samuel Beatty observed that complementary sequences of positive integers occur in another context ([BEATTY]).

Let α be a positive irrational number greater than one and set β so that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. (Algebra shows that β is also an irrational number greater than one.) Set $P_n = \lfloor n\alpha \rfloor$ and $Q_n = \lfloor n\beta \rfloor$. Then:

The sequences $\{P_n\}$ and $\{Q_n\}$ are sure to be complementary.

For example, with $\alpha = \sqrt{2}$ and $\beta = 2 + \sqrt{2}$ we obtain:

$\{P_n\}$: 1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, …
$\{Q_n\}$: 3, 6, 10, 13, 17, 20, …

Establishing the claim is not difficult.

Since α is greater than one, it is clear that no integer is repeated within the sequence $\{P_n\}$. Similarly, no integer is repeated amongst the values of $\{Q_n\}$. But could an integer appear once in each sequence?

Suppose there are integers n and m so that $\lfloor n\alpha \rfloor = \lfloor m\beta \rfloor$. Call this common value k. We then have:

$$k \leq n\alpha < k+1$$
$$k \leq m\beta < k+1$$

(Actually, the inequalities are strict since α and β are each irrational.) Rewriting, we have:

$$\frac{k}{\alpha} < n < \frac{k+1}{\alpha}$$
$$\frac{k}{\beta} < m < \frac{k+1}{\beta}$$

Adding and making use of the relation $\frac{1}{\alpha}+\frac{1}{\beta}=1$ yields the absurdity

$$k < n+m < k+1$$

Thus no integer appears more than once among the two sequences.

Could a positive integer be missing? Suppose integer a is "skipped over" by both sequences $\{P_n\}$ and $\{Q_n\}$. This means we that there must be integers n and m with:

$$n\alpha < a \text{ and } (n+1)\alpha \geq a+1$$
$$m\beta < a \text{ and } (m+1)\beta \geq a+1$$

(Again the inequalities must actually be strict.) Rewriting gives:

$$n < \frac{a}{\alpha} \text{ and } \frac{a+1}{\alpha} < n+1$$
$$m < \frac{a}{\beta} \text{ and } \frac{a+1}{\beta} < m+1$$

and adding yields

$$n+m < a \text{ and } a+1 < n+m+2$$

from which we deduce that a is an integer strictly between $n+m$ and $n+m+1$. This absurdity show that no positive integer can be "missed" by the two sequences.

CHALLENGE: Let γ be a positive irrational number (not necessarily greater than one). Find a formula for the frequency sequence of $\lfloor n\gamma \rfloor$. (Actually … need γ be irrational in your analysis?)

CHALLENGE: In the newsletter we tacitly assumed that no value is repeated infinitely often in the sequence $\{p_n\}$. Would there be a problem if this sequence were eventually constant?

REFERENCES:

[BEATTY]
Beatty, S., "Problem 3173," *American Mathematical Monthly*, **33** (3), (1926), 159.