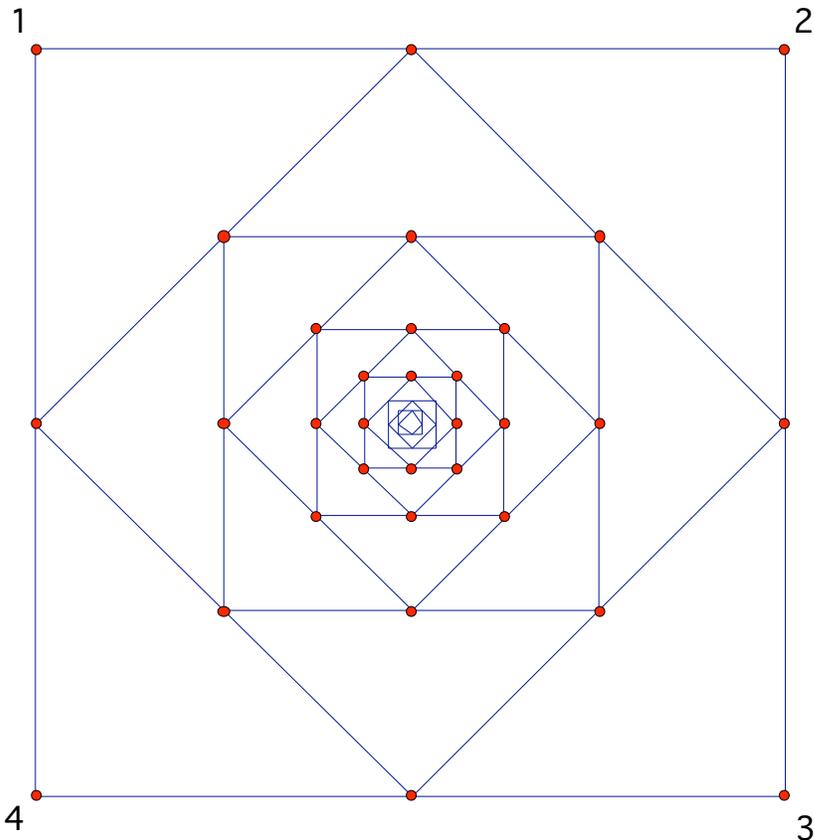


Difference Engine

The numbers 1, 2, 3, and 4 are written at the corners of a large square. At each step, at the midpoint of each side, write the positive (or absolute value of the) difference between the numbers at its ends, so in the diagram below, you would write the numbers 1, 1, 1, and 3 at the midpoints of the sides, forming a new square.



Then repeat that process, writing the positive difference between the endpoints at the midpoint of each side. Eventually, things will get boring...

1. Make a sensible definition of “boring” and determine how many steps it takes for this process to become boring.
2. Using only integers in the range 0 through 1, what’s the longest it can be until this process becomes boring?
3. How about if you can use integers in the range 0 through 2?
4. 0 through 10? 0 through 100?

5. Can you explain how to construct a set of integers that will last for 100 steps or more before becoming boring?
6. Prove that the process always becomes boring if you begin with integers.
7. What if you start with 1, 2, π , 4 instead of 1, 2, 3, 4?
8. How about $1, \sqrt{2}, \pi, \frac{14}{3}$?
9. Does this change your proof? That is, did you assume in your proof that the numbers were all integers? Does the process still always become boring even if the numbers aren't integers? Are there real numbers that will last forever?

Now, what if we use other shapes besides a square, say, an n -sided regular polygon?

10. When $n = 1$ or 2 , if you can even call that a polygon, things get boring really fast. Explain why.
11. When $n = 3$, you may need to change your definition of boring. Why? Do all sequences become boring in the same way?
12. In general, which values of n can use the same definition of boring as the square?
13. Can you establish a general upper limit on how long it will take for a given set of numbers to become boring, based on the largest number in the set and the number of sides of the n -gon?

And what if we add instead of subtracting at each step? Then our old 1, 2, 3, 4 would turn into 3, 5, 7, 5.

14. Do you need a new definition of “boring” or does your old one still work?
15. Do all number patterns at the vertices of a square eventually become boring? Does it work for integers? Fractions? Crazier numbers?
16. What about n -sided polygons?

Difference Engine Teacher's Guide

One path to the solution is through looking at the oddness or evenness of the numbers. No matter what pattern of odds and evens you start with, eventually all the numbers will become even. In fact, the question requiring you to use only 0s and 1s is intended as a hint toward exactly this idea.

Once you have all the numbers even, you could divide all the numbers by 2 and it will still take the same number of steps to reach all 0s, so you can see that the maximum number keeps decreasing. Or, you can say the same thing by saying that after a few more steps, all the numbers will be divisible by 4, and then 8, and then 16, and eventually by such a big power of 2 that the only number they can equal is 0.

Another fruitful path of exploration is to ask what sets of numbers are “equivalent” in terms of their futures, and what sets are equivalent in terms of their pasts. (Perhaps surprisingly, these two questions have different answers!) Maybe this can lead to some ways to “standardize” the numbers so you have fewer than four independent variables.

Ideally students will realize that working backwards is a good strategy, but then get frustrated when they discover that many positions have no predecessor, and then figure out how to fiddle with the position to create a predecessor, but then get frustrated again... For instance, before 1,2,3,4 you might have 1,0,2,5 but there's nothing that can come before that. On the other hand you might have had 2,1,3,6 before 1,2,3,4, and then there are other positions before 2,1,3,6, so you can work backwards for a while. What eventually goes wrong?

Look at what happens if the sequence of numbers is not increasing as you go around the square beginning at the lowest number but instead goes up and down.

With real numbers, things are a little more complicated. By “standardizing” to 0, 1, x , y you can graph the possible positions in the plane, and find a transformation function that says where to go from a given position. Can you find a fixed point in this transformation?

Here's some more assorted comments that might be useful to you in talking about this problem, from a math circle session I did on this problem a long time ago.

In the case of 4 numbers, we looked at even/odd patterns. It was pointed out that due to reflections and rotations, there are only a few cases to check:

EEEE \rightarrow EEEE

OOOO \rightarrow EEEE

OEEE \rightarrow OEE0 \rightarrow OE0E \rightarrow O000 \rightarrow EEEE

E000 \rightarrow OE00 \rightarrow OE0E \rightarrow O000 \rightarrow EEEE

E000 \rightarrow E0E0 \rightarrow O000 \rightarrow EEEE

E0E0 \rightarrow O000 \rightarrow EEEE

So no matter what the even/odd pattern is, it will go to all evens in at most 4 steps.

This is the basis of one good proof, that at this point you could divide them all by two and get a set of four numbers with a smaller maximum, thereby continually decreasing the maximum.

There are also more direct proofs possible showing that the largest number must decrease (immediately if it's not next to a zero, but after a few moves even if it initially is next to a zero.)

We tried a single sequence of 8 numbers, and got the following:

```

1 2 3 4 5 6 7 8
1 1 1 1 1 1 1 7
0 0 0 0 0 0 6 6
0 0 0 0 0 6 0 6
0 0 0 0 6 6 6 6
0 0 0 6 0 0 0 6
0 0 6 6 0 0 6 6
0 6 0 6 0 6 0 6
6 6 6 6 6 6 6 6
0 0 0 0 0 0 0 0

```

Not only did it converge, but if you whack out part of the pattern on the right:

```

      7
     6 6
    6 0 6
   6 6 6 6
  6 0 0 0 6
 6 6 0 0 6 6
6 0 6 0 6 0 6
6 6 6 6 6 6 6 6

```

except for the 7 at the top, it looks surprisingly like Pascal's triangle, except that the 6s (and 7) represent the places in the triangle where there are odd numbers.

Note that rows 1, 3, 7, ..., $2^n - 1$ are all odd numbers.

Thus, it was conjectured (and Pascal's triangle seems to prove) that if you start with a list of 2^n terms, all of which are zero, but with a single non-zero term, the pattern will be exactly as above, and will converge to all zeroes after 2^n steps.

Are there deeper connections to Pascal's triangle here?