Elementary Planar Geometry

What is a geometric solid?

It is the part of space occupied by a physical object.

A geometric solid is separated from the surrounding space by a **surface**.

A part of the surface is separated from an adjacent part by a **line**.

A part of the line is separated from an adjacent part by a point.

A geometric figure **is points, lines, or solids** positioned in a certain way in space.

Examples: Draw a picture of a geometric solid and show the surface, lines and points in this solid.

Can you draw pictures of geometric solids bounded by one, two, three, four planes?

We say two geometric figures are **congruent** if by moving one of the figures it is possible to superimpose it onto the other so that the two figures become identified with each other in all their parts.

Example: Draw a picture of two geometric solids which are congruent to each other.

Draw a picture of two congruent surfaces.

Draw a picture of two lines which are congruent and a picture of two lines which are not congruent. Can you make a guess when are two lines congruent?

If a geometric figure is congruent to another geometric figure, which is in turn congruent to a third geometric figure, then the first geometric figure is congruent to the third. Can you explain why?

The plane

We are most familiar with a special surface. This surface is **flat** and is called a **plane**. What comes to your mind when you think of a plane? For example, a window pane, or the water surface in a calm pond.

A plane has a special property: We can superimpose a plane on itself or any other plane in a way that takes one given point to any other point. We can also do this after flipping the plane upside down.

Can you explain the corresponding symmetries?

Can you give an example if a surface other that the plane which, like the plane, can be superimposed on itself in a way that takes any one point to any other given point?

What about an example of a surface in which any one point cannot be taken to any other given point?

The straight line

The straight line is the simplest of lines. We can imagine it as a ray of light or a very thin wire. The most fundamental property of a straight line is the following: *For every two points is space, there is a straight line passing through them and such a line is unique.*

This property gives us other properties:

If two straight lines are aligned with each other in such a way that two points of one line coincide with two points of the other, then the lines coincide in all their other points as well.

And

Two straight lines can intersect at most at one point.

Can you give reasons why these properties follow from the fundamental property?

A plane not containing a straight line can intersect it at most at one point. Can you explain why?

Can you explain whether a plane and a line must intersect?

A straight line can lie in a plane. If a straight line passes through two points of a plane, then all the points of this line must lie in this plane. Why do you think is this true?

For two points in a plane, there is a straight line *lying in this plane* and passing through these two points. Can you explain why this is true? Moreover, we already know that such a line must be unique.

Can you think of surfaces other than the plane which, like the plane, together with each point lying on the surface contain a straight line passing through this point?

The unbounded straight line, Ray and line segments

If a straight line extends indefinitely on both sides we call it an **infinite (or unbounded)** straight line. A straight line is denoted by two uppercase letters marking any two points on it.

A part of a straight line bounded on both sides is called a straight segment (or a **segment**).

A straight line which terminates in one direction only is called **a ray (or a half-line).**

Congruent and non-congruent segments

Two segments are congruent if they can be laid one onto the other so that their endpoints coincide.

We can use a compass to mark a line segment congruent to a given segment.

On a given line, mark a segment congruent to four times a given segment, using a compass as few times as possible.

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The Circle

A circle is a set of points which is at the same distance from a special point called the center. The distance of any point on the circle from the center is called the radius of the circle.

A line intersecting the circle at any two points is called a **secant.**

A segment both of whose endpoints lie on the circle is called a **chord.**

A chord passing through the center is called a **diameter.**

A part of a circle contained between any two points on the circle is called an **arc.**

Draw examples of all of these on the following circle.

Two arcs of the same circle (or two congruent circles) are congruent if they can be aligned so that their endpoints coincide. In the picture above, can you draw an arc which is congruent to the arc you have already drawn?

Angles

A figure formed by two rays drawn from the same point is called an **angle.** The rays which form the angle are called its **sides,** and their common endpoint is called the **vertex** of the angle.

The sides of an angle divide the whole plane into two regions: the **interior** and the **exterior.**

Draw a picture of an acute angle indicating the interior and exterior regions. How about an obtuse angle? How about an angle larger than 180°?

Two angles are considered **congruent** if by moving one of them it is possible to identify it with the other.

We can add angles by first identifying a ray of the first angle with a ray of the second angle and then by aligning the two angles with this identification. Let us look at an example.

Similarly we can subtract angles, multiply and divide angles by a whole number.

A ray which divides a given angle into halves is called the **bisector** of the angle.

Two straight lines intersect at an angle containing 25°. Draw a picture showing this. Then find the measures of the remaining three angles formed by these lines.

Three lines passing through the same point divide the plane into six angles. Two of them turned out to contain 25° and 55°. Draw a picture showing this. Find the measures of the remaining angles.

Suppose we are given a 19° arc on a circle. Can you show how to construct a 1° arc on the same circle using only a compass?

Perpendicular lines

A **right angle** is an angle of 90°. A **straight angle** is an angle of 180°. An angle smaller than the right one is called **acute.** An angle greater than right but smaller than straight is called **obtuse.** Here are examples showing these various angles:

What is the smallest number of acute angles which add up to the full angle?

Two angles are called **supplementary** if they have one common side, and their remaining two sides form continuations of each other. Since the sum of such angles is a straight angle, *the sum of two supplementary angles is 180°.*

Draw a picture of supplementary angles.

Find the measure of an angle which is congruent to twice its supplementary one.

Bisectors of two supplementary angles are perpendicular to each other. Explain why this is true.

If one of the four angles formed by two intersecting lines is right, then the other three angles must be right as well. Illustrate this by drawing a picture.

Can you show why the following is true: From any point lying outside a given line one can drop a perpendicular to this line, and that such perpendicular is unique.

Polygons and triangles

Straight segments not lying on the same line are said to form **a broken line** if the endpoint of the first segment is the beginning of the second one, the endpoint of the second one is the beginning of the third one, and so on.

These segments are called **sides**, and the vertices of the angles formed by the adjacent segments **vertices** of the broken line. Here is an example:

A broken line is called **convex** if it lies on one side of each of its segments continued indefinitely in both directions. Draw a picture of a broken line which is convex and another which is not.

A broken line is **closed** if its endpoints coincide. A closed broken line may have self-intersections. Here is an example:

The figure formed by a non-self-intersecting closed broken line together with the part of the plane bounded by this line is called a **polygon.**

The sides and vertices of this broken line are called respectively **sides** and **vertices** of the polygon, and the angles formed by each two adjacent sides (**interior) angles** of the polygon.

A polygon is called **convex** if it is bounded by a convex broken line.

Four points on the plane are vertices of three different quadrilaterals. How can this happen?

Can a convex broken line self-intersect?

Is it possible to tile the entire plane by non-overlapping polygons all of whose angles contain 140° each?

Each diagonal of a quadrilateral either lies entirely in its interior, or entirely in its exterior. Explain why this is true. Give an example of a pentagon for which this is false.

How many intersection points can three straight lines have?

A triangle is called **isosceles** when two of its sides are congruent, and equilateral when all three sides are congruent.

A triangle is called **acute** when all of its angles are acute, and **right** when one of the angles is right. It is called **obtuse** when one of the angles is obtuse.

In a right triangle, the sides of the right angle are called **legs** and the side opposite to the right angle the **hypotenuse.**

Can you draw pictures showing various kinds of triangles?

One of the triangle's sides is referred to as the **base**, in which case the opposite vertex is called *the* vertex of the triangle, and the other two sides are called lateral.

The perpendicular dropped from the vertex to the base or its continuation is called an **altitude.**

Can you explain why in a right triangle, three altitudes must pass through a common point?

The segment joining the vertex of a triangle with the midpoint of the base is called a **median.** The segment dividing the angle at the vertex into halves is called a **bisector** of the triangle.

In any triangle, two medians must intersect. Why is this true? What about any two bisectors? altitudes?

Give an example of a triangle such that only one of its altitudes lies in its interior.