FOLDING and POURING

This is a sample (draft) chapter from:

MATHEMATICAL OUTPOURINGS

Newsletters and Musings from the St. Mark's Institute of Mathematics

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This material was – and can still be – used as the basis of a successful MATH CIRCLE activity.

The St. Mark's
Institute of Mathematics
Newsletter

JANUARY 2009

THIS MONTH'S PUZZLER:
One gallon of water is distributed between two containers labeled A and B. Three-quarters of the contents of A are poured into B, and then half the contents of B are poured back into A.

This process of alternately pouring from A to B (three-quarters of the content) and then from B to A (half the content) is repeated indefinitely.

What happens in the long run?

TODAY'S TIDBIT: PAPER FOLDING
Here's something to try:

Take a strip of paper and make a crease mark at some arbitrary position.
Make a new crease half-way between this position and the left end of the strip by folding the left end of the paper.
Make a new crease half-way between this new mark and the right end of the strip by folding the right end of the paper.
Repeat alternating left and right folds, with each fold made to the most recent crease mark.
Notice that the system seems to "converge" to two positions on the strip. What are these two positions?

To answer …. Suppose the strip is 1 unit long and the first crease is at position x. Then a left fold creates a new crease at position

$$x \mapsto \frac{x}{2}$$

and a right fold a crease at position

$$x \mapsto x + \frac{1-x}{2} = \frac{1}{2} + \frac{x}{2}$$

These transformations have particularly nice interpretations if we write x as a "decimal" in base two.

Aside on Base Two: In ordinary base ten arithmetic the decimal $0.abcd\ldots$ represents the quantity $\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \frac{d}{10000} + \cdots$. In base two $0.abcd\ldots$ represents $\frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \cdots$.
Every number has a base two representation. (Is this at all obvious?) For example, $3/4 = 0.11$, $1/3 = 0.010101\ldots$ and $0.11111\ldots = 1$. (Why? Read on!) Notice that if

$$x = \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \cdots = 0.abcd\ldots$$

then

$$2x = a + \frac{b}{2} + \frac{c}{4} + \frac{d}{8} + \cdots = a.bcd\ldots$$

and

$$\frac{x}{2} = \frac{a}{4} + \frac{b}{8} + \frac{c}{16} + \frac{d}{32} + \cdots = 0.0abcd\ldots$$

This shows that *multiplying and dividing by two shifts the decimal point.*

So … To evaluate $w = 0.111111\ldots$, say, observe that

$$2w = 1.1111\ldots$$
$$= 1 + 0.111\ldots = 1 + w$$

giving $w = 1$. Also, if $z = 0.010101\ldots$ then $2z + z = 0.11111\ldots = 1$ yielding $z = 1/3$.

Exercise: What is the value of $0.001001001\ldots$ in base two? What is $0.090909\ldots$ in base 10?

Now back to paper folding …

If the initial crease is at $x = 0.abcd\ldots$ then a left fold produces a new crease at

$$x/2 = 0.0abcd\ldots$$

and a right fold at

$$\frac{1}{2} + \frac{x}{2} = 0.1 + 0.0abcd\ldots = 0.1abcd\ldots$$

These actions simply insert a 0 or a 1 in first slot of the binary representation of x. Thus if we make four right and left folds, we'll have a crease at position $0.0101abcd\ldots$, or with ten folds at position $0.0101010101abcd\ldots$ and so on. After more folds we obtain creases closer and closer to position $0.01010101\ldots = 1/3$ and to position $0.101010101\ldots = 2/3$. (Did you see this on your paper?) Notice that the location of the initial fold is irrelevant – the folding exercise always converges to these two locations.

Comment: Paper folding is equivalent to a water-transfer problem: *One gallon of water is distributed between two containers labeled A and B. Half the contents of A are poured into B (changing contents o f A via* $x \mapsto \frac{x}{2}$*) and then half the contents of B are poured back into A (changing contents of A via* $x \mapsto x + \frac{1-x}{2} = \frac{1}{2} + \frac{x}{2}$*). This process of alternately pouring half from A to B and then half from B to A is repeated indefinitely. What happens in the long run?*
Answer: The system approaches the same one-third/two-third oscillation.

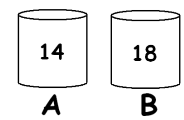
We can play with this. Instead of transferring half the contents according to the pattern $A \to B,\ B \to A$ what if we transferred half following the cycle $A \to B, A \to B, B \to A$? (This corresponds to transferring three-quarters of the contents of A to B and bringing half of the contents back. This is the opening puzzler of the newsletter.) Do you see that this situation converges to the state of having $0.100100100\ldots = 4/7$ gallons in container A at the start of each cycle (followed by 2/7, then 1/7, and back to 4/7)?

Another variation: Suppose we transfer $\frac{9}{10}$ of the liquid back and forth between the two containers according to $A \to B,\ B \to A$. The transformation rules that arise

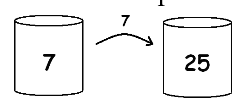
$$x \mapsto \frac{x}{10} \qquad x \mapsto \frac{9}{10} + \frac{x}{10}$$

have nice interpretations in base 10 decimals. Can you see that this game "converges" to the oscillatory state of $0.090909\ldots = 1/11$ and $0.909090\ldots = 10/11$ gallons?

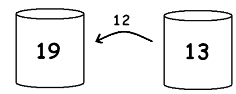
RESEARCH CORNER:
Consider a discrete version of this game. Suppose we have 14 marbles in one cup labeled A and 18 marbles in a second cup labeled B.



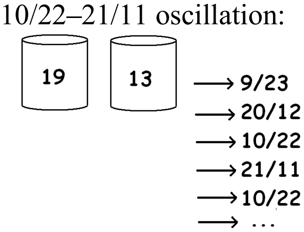
Pour half the contents of cup A into B…



… and then half the contents of cup B back into A keeping the extra odd marble in B.



Let's make the rule: *Cup B should always keep or be given any extra odd marble.* Repeatedly pouring half (or just over half in the odd case) of the contents of A into B and then half (or just under half in odd case) of the contents of B into A enters a 10/22–21/11 oscillation:



a) Does every initial distribution of 32 marbles lead to a 10/22 – 21/11 oscillation?
b) Prove that if the number of marbles with which we start is a power of two, then every initial distribution enters the same final oscillation. Show that this is not the case for the 9 marble game.
d) Is there a general theory about the types of oscillations into which games will converge? Does there exist a game that enters a cycle of period different from two?
e) How does all of the above change if we handle the "odd marbles" in a different way?
f) What if we transferred fractions different from one half?

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FOLDING and POURING
January 2009

COMMENTARY, SOLUTIONS and THOUGHTS

Folding dyadic fractions made a strong appearance in the March 2005 newsletter. Their reappearance here in connection with water pouring is nice.

Let's examine pouring on a little more detail. Rather than ask what results from performing a prescribed series of pouring actions, let's ask if we can find a set of pouring actions that yield a prescribed result!

Here's the set-up:

> We have of one gallon of water distributed between two containers A and B, starting with x gallons in A and $1-x$ gallons in B. We have a fixed number r between 0 and 1, and we are allowed to pour that proportion of the contents of A into B, or vice versa.

We ask:

> Given a value $\alpha \in (0,1)$, is there a sequence of pouring moves we could perform that would yield α gallons of water in container A up to any prescribed degree of accuracy?

Pouring half: $r = \frac{1}{2}$

As we saw in the newsletter, if we write $x = 0.abcd...$ as a binary "decimal," the act of pouring half the water from container A to B leaves $\frac{x}{2} = 0.0abcd...$ gallons in A (insert "0"), and the act of pouring half the contents of B into A leaves $\frac{x}{2} + \frac{1}{2} = 0.1abcd...$ gallons in A (insert "1"). If we approximate α by truncating its binary decimal expansion at the nth decimal place and follow the sequence of pouring instructions given by those first n digits after the decimal point, we obtain an amount of water in container A that matches α in those n places. This quantity differs from α gallons by no more than $\frac{1}{2^{n+1}}$ gallons, which can be made arbitrarily small by choosing sufficiently large n. (Of course, if α is a dyadic fraction, we can match this amount precisely.)

Pouring More than Half: $r = \frac{2}{3}$

Suppose we now pour two-thirds of the water from container one container to the other at each move. If there are x gallons in container A, pouring from A to B leaves $\frac{x}{3}$ gallons in this container, and pouring from B to A produces $x + \frac{2}{3}(1-x) = \frac{x}{3} + \frac{2}{3}$ gallons in A. If we write x as a "decimal" in base 3, $x = 0.abcd...$ these two moves correspond to the actions:

$$0.abcd... \mapsto 0.0abcd...$$
$$0.abcd... \mapsto 0.2abcd...$$

Thus we can approximate any quantity α within container A arbitrarily closely provided that α has a ternary expansion involving only the digits 0 and 2. That is, we can approach any desired value in Cantor's middle-thirds set but we cannot reach those values that are not in that set! (For instance, we will never see half a gallon of water in container A with close accuracy, unless we happen to start with that amount in A!)

CHALLENGE: Let r be any value between $\frac{1}{2}$ and 1. Write $r = 1 - \frac{1}{b}$ with $b > 2$. Then a pouring action changes the amount x of water in container A to either $\frac{x}{b}$ gallons or to $\frac{x}{b} + \frac{b-1}{b}$ gallons. Thinking "decimals base b" (even if b is not an integer?) what can you say about which quantities α can be well approximated as an amount of water in container A?

Pouring Less than Half: $r = \frac{1}{3}$

Suppose we now pour one-third of the water from container one container to the other at each move. If there are x gallons in container A, pouring from A to B leaves $\frac{2}{3}x = \frac{x}{3/2}$ gallons in this container, and pouring from B to A produces $x + \frac{1}{3}(1-x) = \frac{x}{3/2} + \frac{1/2}{3/2}$ gallons in A. If we write x as a "decimal" in base $1\frac{1}{2}$ using the "digits" 0 and $1/2$, $x = 0.abcd...$, these two moves correspond to the actions:

$$0.abcd\ldots \mapsto 0.0abcd\ldots$$

$$0.abcd\ldots \mapsto 0.\frac{1}{2}abcd\ldots$$

COMMENT: This is mighty strange! By the "decimal $0.abcd\ldots$ in base one-and-a-half" we really mean the number in the interval $[0,1]$ given by the sum $a\left(\frac{2}{3}\right)+b\left(\frac{2}{3}\right)^2+c\left(\frac{2}{3}\right)^3+d\left(\frac{2}{3}\right)^4+\cdots$ with each coefficient being either 0 or $\frac{1}{2}$. Any number x in the unit interval can be approximated arbitrarily closely by a finite decimal in base one-and-a-half:

If $\frac{1}{2}\left(\frac{2}{3}\right)$ is smaller than x, choose $a=\frac{1}{2}$, otherwise set $a=0$.

If $\frac{1}{2}\left(\frac{2}{3}\right)^2$ is smaller than $x-a\left(\frac{2}{3}\right)$, choose $b=\frac{1}{2}$, otherwise set $b=0$.

If $\frac{1}{2}\left(\frac{2}{3}\right)^3$ is smaller than $x-a\left(\frac{2}{3}\right)-b\left(\frac{2}{3}\right)^2$ choose $c=\frac{1}{2}$, otherwise set $c=0$, and so on.

If we follow this procedure n times, then the sum of those first n terms differ from x by no more than $\frac{1}{2}\left(\frac{2}{3}\right)^{n+1}+\frac{1}{2}\left(\frac{2}{3}\right)^{n+2}+\cdots=\frac{1}{2}\left(\frac{2}{3}\right)^{n+1}\cdot 3=\left(\frac{2}{3}\right)^n$ which can be made arbitrarily small.

QUESTION: I was careful here to avoid the issue of writing x as in infinite decimal. In what sense can this construct be followed to an infinite degree? Does every quantity in $[0,1]$ have a (well defined?) base one-and-a-half decimal expansion?

For any desired amount α gallons of water in container A we now see that there is a sequence of pouring moves that produces that amount in container A up to any prescribed degree of accuracy. (Write α in base one-and-a-half, up to at least n decimal places, and follow the sequence of pouring moves they dictate.)

CHALLENGE: Let r be any value between 0 and $\frac{1}{2}$. Prove that we can approximate any amount $\alpha\in(0,1)$ gallons of water in container A to any prescribed degree of accuracy, with no restriction on the value of α

These challenges are discussed and solved in [IGA] with good care. (Though I do think it is exciting to think of terms of decimal expansions all the way through!)

DISCRETE POURING

We can answer challenge b) posed in the research corner relatively easily:

> *Two containers* A *and* B *contain a total of* 2^n *marbles between them. We pour half the marbles from* A *into* B *(taking the extra marble as well into* B *if we are dealing with an odd count) and then half the marbles from* B *into* A *(leaving the extra marble in* B *if we are dealing with an odd count). These pouring actions from* A *to* B *and then from* B *to* A *are repeated indefinitely. What happens in the long run?*

If there are x marbles in container A, pouring from A to B leaves $\left\lfloor \frac{x}{2} \right\rfloor$ in that container (where these square brackets mean "round down to the nearest integer") and a little thought shows that pouring the contents from B to A yields $\left\lfloor \frac{x}{2} \right\rfloor + 2^{n-1}$ marbles in container A. If we write x as an n-digit number in base two, $x = abc....de$, then these operations correspond to deleting the last digit and inserting a 0 and a 1, respectively, in the leftmost position:

$$x = abc....de \to \left\lfloor \frac{x}{2} \right\rfloor = 0abc\ldots d$$

$$x = abc....de \to \left\lfloor \frac{x}{2} \right\rfloor + 2^{n-1} = 1abc\ldots d$$

Thus, eventually, the contents of container A is sure to oscillate between just two values: $1010101.... = \left\lfloor \frac{2}{3}x \right\rfloor$ and $0101010... = \left\lfloor \frac{1}{3}x \right\rfloor$.

REFERENCES:

[IGA]
Iga, K., "The truck driver's straw problem and Cantor sets," *College Mathematics Journal*, **39** (2008), 280-290.