Fibonacci Numbers

David Patrick
patrick@artofproblemsolving.com

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The **Fibonacci numbers** are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

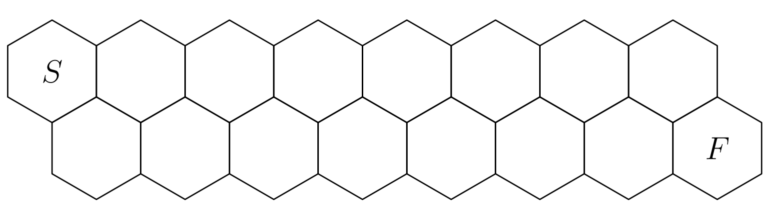
where each number (after the first two 1's) is the sum of the previous two numbers.

By convention, we let F_n denote the n^{th} Fibonacci number. So $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, and so on. This means that $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. We sometimes also designate $F_0 = 0$. (Note that this also makes $F_2 = F_1 + F_0$ true.)

1. In how many ways can a person climb a flight of n stairs, if on each step the person climbs 1 or 2 stairs?
2. In how many ways can we pick integers from the set $\{1, 2, \ldots, n\}$, if we're not allowed to pick two consecutive integers?
3. In how many ways can a $2 \times n$ checkerboard be tiled with 1×2 tiles?
4. What is the sum of the first k odd-position Fibonacci numbers? (For example, if $k = 3$, then we get $1 + 2 + 5 = 8$.)
5. What is the sum of the squares of the first k Fibonacci numbers? (For example, if $k = 3$, then we get $1^2 + 1^2 + 2^2 = 6$.)
6. Which Fibonacci numbers are even and which are odd? Which Fibonacci numbers are multiples of 3? Multiples of 5? Can you generalize?

The next group of problems was suggested by James Tanton.

For the next group of problems, we'll consider a honeycomb grid of hexagons with 2 rows and n hexagons in each row. For example, the grid below is for $n = 8$:



We wish to count paths from S to F in which each step is to a hex immediately adjacent on the right. For example, below is one such path:

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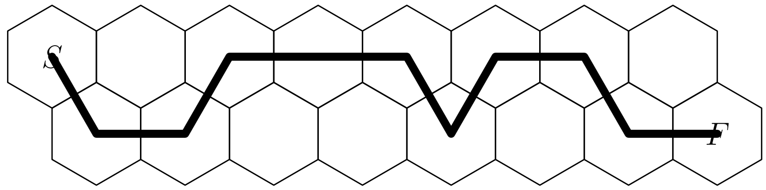

7. How many such paths are there? (If you're stuck on this, try the next problem first and then come back to this one.)

8. How many paths (starting at S) to each individual hex in the diagram?

9. Use the honeycomb diagram to prove that $F_aF_b + F_{a+1}F_{b+1} = F_{a+b+1}$ for any positive integers a and b.

10. (a) There are 8 ordered partitions of the number 6 into odd parts:

    $5+1 = 1+5 = 3+3 = 3+1+1+1 = 1+3+1+1 = 1+1+3+1 = 1+1+1+3 = 1+1+1+1+1+1.$

    How many odd ordered partitions are there of an arbitrary number n?

    (b) There are 5 ordered partitions of the number 6 into numbers larger than 1:

    $$6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2.$$

    How many such ordered partitions are there of an arbitrary number n?

11. The language on the island of Abeeba consists of only the letters A, B, and E. Any combination of letters is a word in the language, **except** that an A can never be immediately followed by an E. (For example, AABEA is a valid word, but ABBAEB is not.) How many n-letter words are there in this language?

Some further resources:

- The book *Intermediate Counting & Probability* by David Patrick, specifically Chapter 9, "Fibonacci Numbers."
- www.jamestanton.com, in particular the document "Fibonacci Surprises."
- Dr. Ron Knott's website on Fibonacci numbers at:
  http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html

Other Special Numbers

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We have 4 problems that (at first) seem unrelated. The goal is to see how the problems are all related. As you work through them, compare your answers to the different problems.

1. Imagine an even number of people standing around a circle. We wish to investigate how many ways they can simultaneously pair up and shake hands, so that no two people cross arms.

   (a) Why do we need an even number of people?

   (b) What's the answer for 2 people?

   (c) What's the answer for 4 people?

   (d) What's the answer for 6 people? (If you can find 5 other people, you might be able to experiment!)

   (e) What's the answer for 8 people? (Now the problem is big enough that experimentation might not work so well. Can you find a more clever way to approach the problem, using the answers that you already have worked out for smaller numbers of people?)

   (f) Can you generalize?

2. We want to count the number of ways that we can validly arrange parentheses. A "valid" arrangement is one that, when reading left-to-right, we never have more)'s than ('s. For example, ((()())()) is valid, but (()()))(() is invalid.

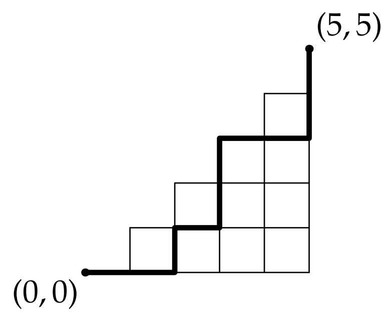
   (a) In how many ways can we validly arrange 1 set of parentheses?

   (b) 2 sets?

   (c) 3 sets? (Try listing them all.)

   (d) Can you generalize?

3. In how many ways can we walk from $(0,0)$ to (n,n) (where n is a positive integer), always taking unit steps up or to the right, so that we never pass through a point (x,y) with $y > x$? For example, a path for $n = 5$ is shown to the right.



   (a) Count the number of paths for $n = 1$, $n = 2$, and $n = 3$. (You can use the attached sheet to try to draw some of them.)

   (b) How is this related to the parentheses problem above? Is there a straightforward way to convert a path to a valid parentheses arrangement and vice versa?

   (c) Are there any "real-world" problems that you can think of that are modeled by this problem? (Think sports.)

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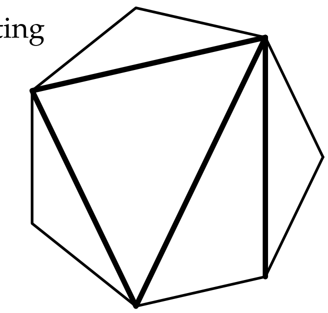
4. *Triangulating* a regular polygon means to draw enough non-intersecting diagonals to divide the polygon into a bunch of triangles. An example of a triangulation of a heptagon is shown to the right. We wish to count how many ways we can triangulate a regular n-sided polygon.



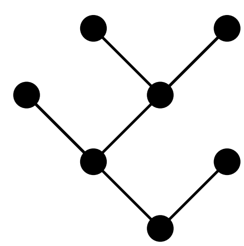
   (a) In how many ways can we triangulate a triangle?
   (b) How about a square?
   (c) How about a pentagon? (You can try to draw some of them on the attached sheet.)
   (d) Can you generalize?

The numbers that you should get in problems 1–4 are called *Catalan numbers* (in honor of the 19^{th}-century Belgian mathematician Eugène Catalan). MIT math professor Richard Stanley has listed on his website 198 different counting problems that are solved by the Catalan numbers! Here are a few more to play with:

5. In how many ways can we place n indistinguishable balls into boxes B_1, B_2, up through B_n, such that boxes B_1 through B_i have a total of no more than i balls (for all $1 \le i \le n$)?

6. In how many ways can we place several identical coins in one or more rows on a flat surface, such that there are n coins in a row in the bottom row, and each coin (above the bottom row) touches the two coins directly beneath it?

7. How many rooted binary trees are there with n internal vertices? (A *binary tree* is a graph with a *root* at the bottom, in which each vertex has exactly 0 or 2 upward branches, and in which there are no loops. A vertex with 2 branches is an *internal vertex*, and a vertex with 0 neighbors is a *leaf*. An example for $n = 3$ is shown to the right—the root is the vertex at the bottom.)



Now for the generalization:

8. Can you find a recursive formula for the Catalan numbers? This is a formula in which we write the n^{th} Catalan number in terms of smaller Catalan numbers.

9. Can you find a closed formula for the Catalan numbers? This is a formula in which we write the n^{th} Catalan number in terms of n. The closed formula is pretty hard. There is a really clever way to do this using problem 3 above. As a hint, try to compare the n^{th} Catalan number to the binominal coefficient $\binom{2n}{n}$.

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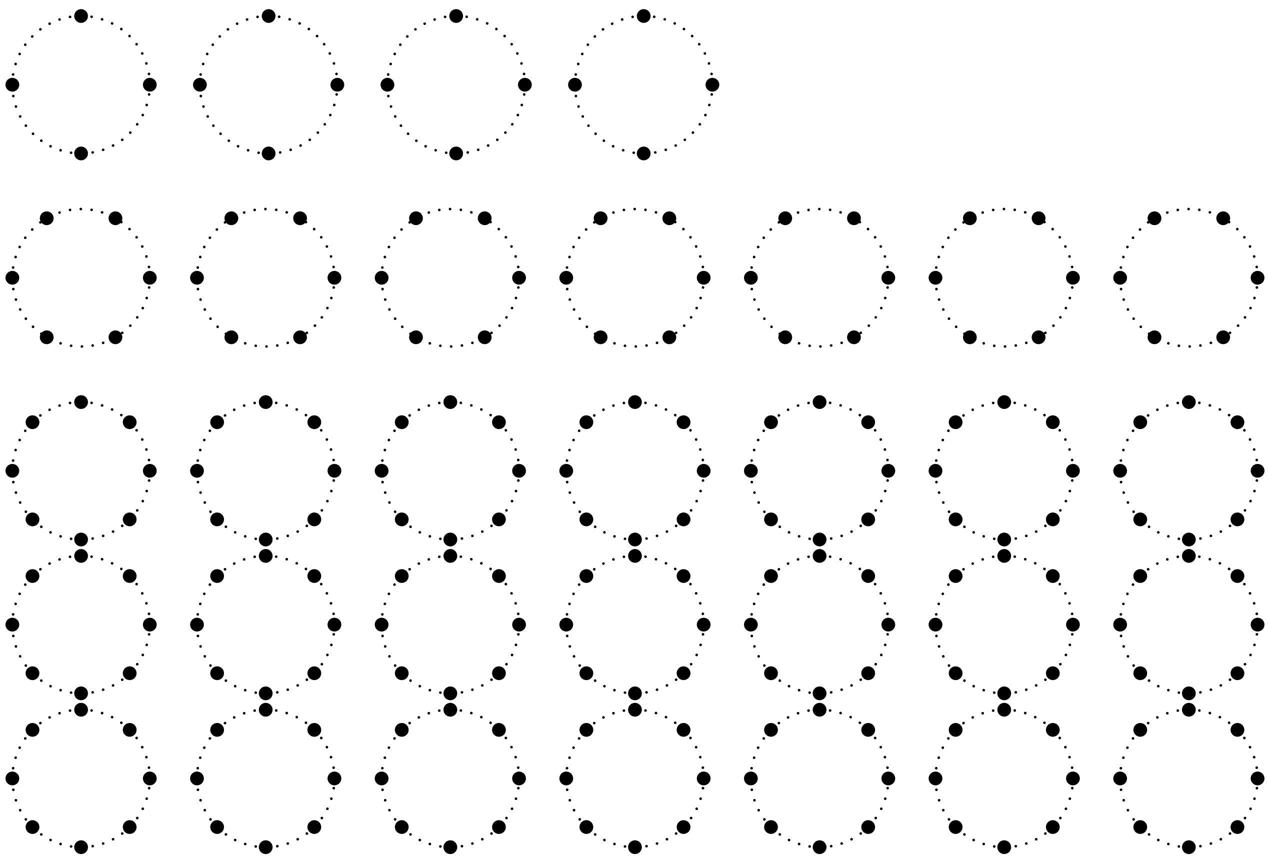
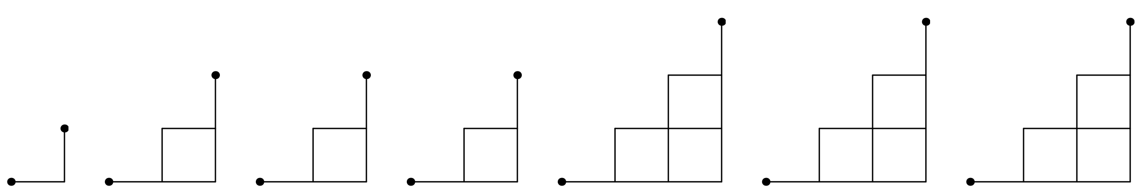
Diagrams for Problem 1: the dots are people, draw lines to indicate handshakes. We've given you extras in case you make a mistake: you shouldn't need them all to draw all the possibilties.



Diagrams for Problem 3: We've given you extras in case you make a mistake: you shouldn't need them all to draw all the possibilties.



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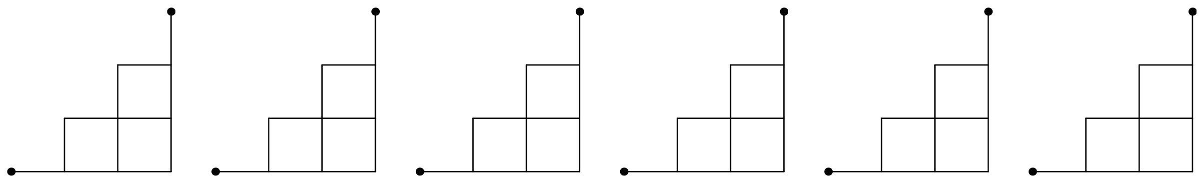
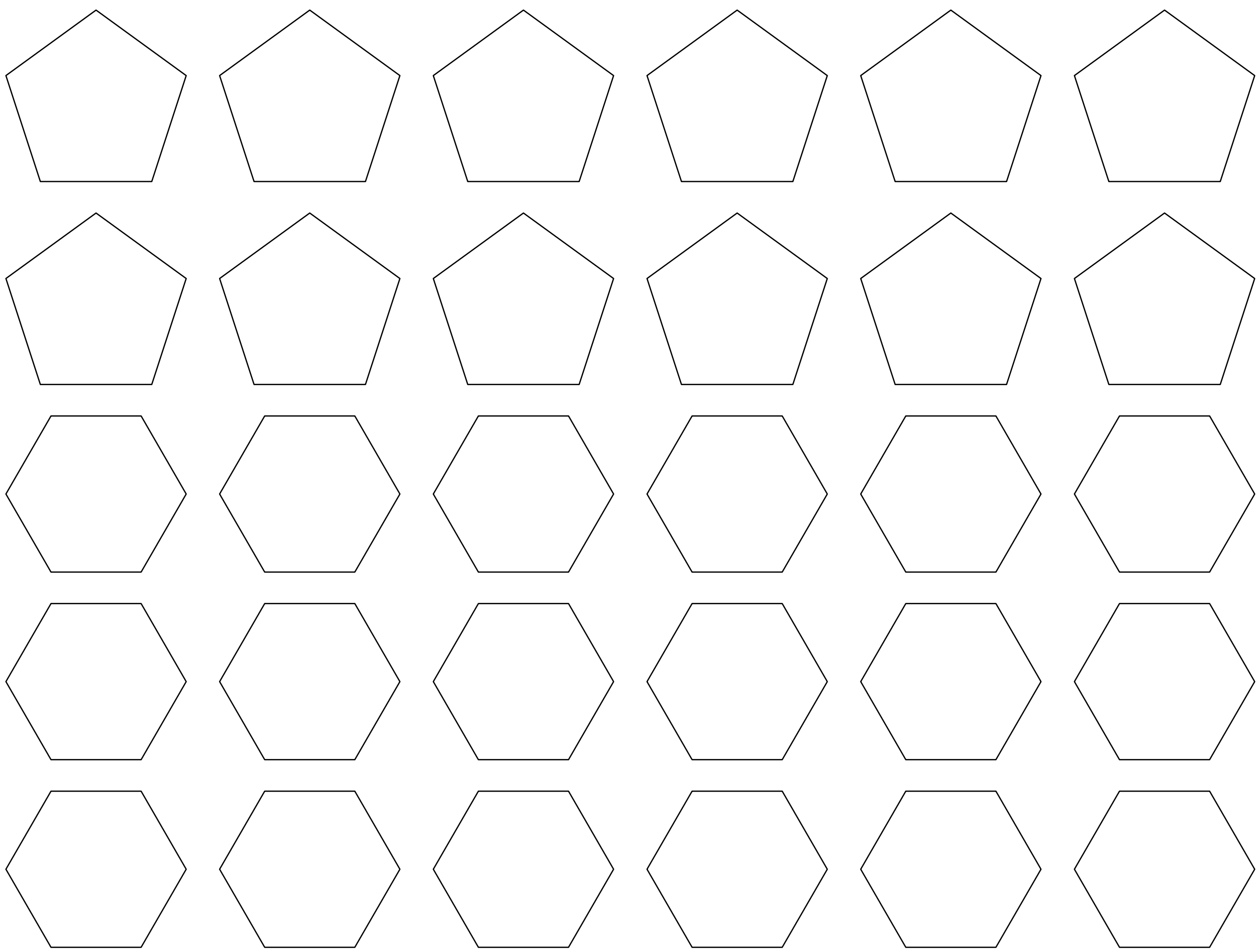


Diagrams for Problem 4: Here are some regular polygons to try to triangulate. Again, there are extras here: you don't need to use them all to draw all the possibilities.



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Notes for instructors

The Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

These are indexed starting at $n = 0$ (so $C_0 = C_1 = 1$, $C_2 = 2$, etc.)

The recursive formula is

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 = \sum_{k=0}^{n-1} C_kC_{n-1-k}.$$

Problems 1 and 2 naturally lead to this formula. For instance, in problem 2, delete the first left parenthesis and its matching right parenthesis, and this breaks the remainder into two smaller sub-problems who "sizes" sum to $n-1$. In problem 1, pick one person to be "special," then that person's handshake splits the group into two smaller groups that sum to $2(n-1)$ people.

The closed formula is

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{1} = \frac{(2n)!}{(n+1)!n!}.$$

This can be obtained from problem 3: "illegal" paths to (n, n) (that cross above $y = x$) can be reflected at the first point for which $y > x$, and this gives a path to $(n-1, n+1)$. (See Problem 10.14 in [1] for more detail.)

The answer is C_n for each of problems 1–3 (where in problem 1 there are $2n$ people), and C_{n-2} for problem 4 (triangulating an n-gon).

Problems 2 and 3 naturally correspond: a move to the right is a (, and a move up is a). Problems 1 corresponds to 2 as well: each pair of people shaking hands is a matching set of parentheses.

More details and further problems in sections 10.5 and 10.6 of [1]. Stanley's web page (with 198 different ways to count them) is [2].

References:

[1] D. Patrick, *Intermediate Counting & Probability*, AoPS Inc., 2007.

[2] R. Stanley, "Information on *Enumerative Combinatorics*," `http://math.mit.edu/~rstan/ec/`.