Fractions Revisited

Joshua Zucker, August 26, 2010
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1. *Fractions Quick Concept Quiz* (Thanks to Jason Dyer's "The Number Warrior" blog)
   i. Why can every fraction be represented an infinite number of ways?
   ii. How do you simplify a fraction? This procedure is often called "reducing" – why is this a good name? Why is this a bad name?
   iii. When is simplifying fractions important? When is it not important?
   iv. Why do the denominators need to be the same when adding (or subtracting) fractions?
   v. Why do the denominators **not** need to be the same when multiplying two fractions?
   vi. Why is $\frac{2}{3}+\frac{5}{3}=\frac{7}{6}$ wrong and $\frac{2}{3}\times\frac{5}{3}=\frac{10}{9}$ right?
   vii. Why is dividing by 1/2 the same as multiplying by 2?
   viii. How can common denominators help in dividing fractions?

2. *Smallest Sum* (From Sam Vandervelde)
   Using the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try 99/96 + 97/98 but that is not the smallest sum.

3. *Simpsons* (with thanks to http://www.cut-the-knot.com)
   i. Lisa's lemonade stand sells 20 cups of lemonade for 30 cents each and 80 cookies for 50 cents each. What is the average price per item?
   ii. Bart's lemonade stand sells 80 cups of lemonade for 40 cents each and 20 cookies for 60 cents each. What is the average price per item?
   iii. Whose lemonade stand has the lower prices? Per item? Per cup? Per cookie?

4. *More Simpsons*
   i. In their first basketball practice, Bart makes 5 out of 11 free throws while Lisa makes 3 out of 7. Who is the better free throw shooter?
   ii. In their second basketball practice, Bart makes 6 out of 9 free throws while Lisa makes 9 out of 14. Who is the better free throw shooter?
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5. *In the Space Between*
   i. Name a fraction between 11/15 and 7/10.
   ii. Name the fraction with smallest denominator between 11/15 and 7/10.
   iii. (Adapted from davidbau.com, which in turn adapted it from Gelfand and Shen *Algebra*) First you draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into 7+13 = 20 equal pieces. How many marks are on each piece?

Fractions Revisited - Leader Notes

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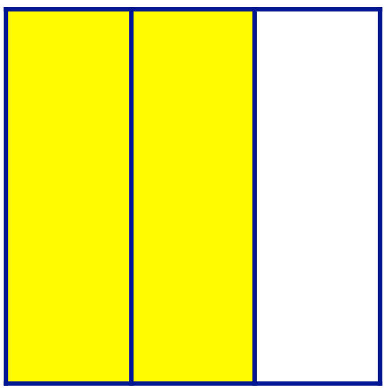
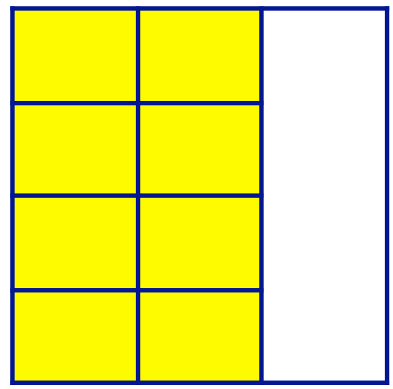
1. *Fractions Quick Concept Quiz* (Thanks to Jason Dyer's "The Number Warrior" blog)

   i. Why can every fraction be represented an infinite number of ways?
   The Key Idea to communicate here is that a fraction represents a number of pieces of a certain size. Subdividing the pieces, so you have *n* times as many pieces that are *n* times smaller, gives another representation. This also points toward representing fractions with pieces of a unit square or rectangle, rather than a circle: it's so easy to subdivide perpendicularly to the existing pieces!



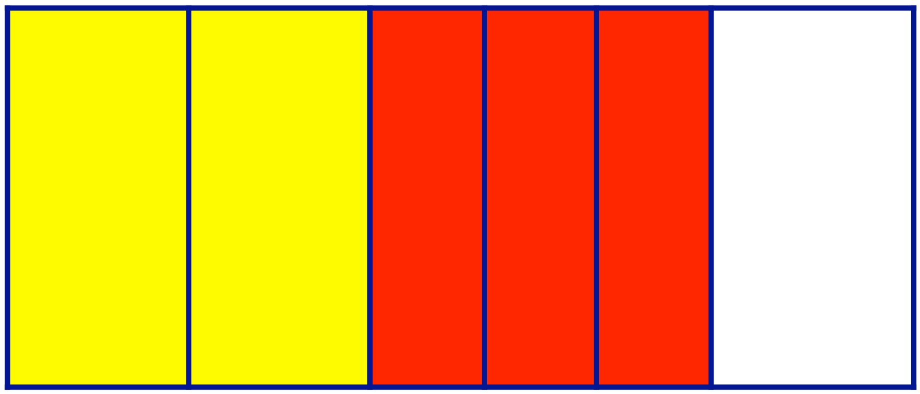


   ii. How do you simplify a fraction? This procedure is often called "reducing" – why is this a good name? Why is this a bad name?
   Terminology is definitely better with "simplifying" than "reducing", but it is good to reinforce that we are reducing the numerator (number of pieces) and the denominator (number of pieces that make up a unit), but increasing the size of the pieces, and of course most importantly leaving the numerical value of the fraction unchanged.

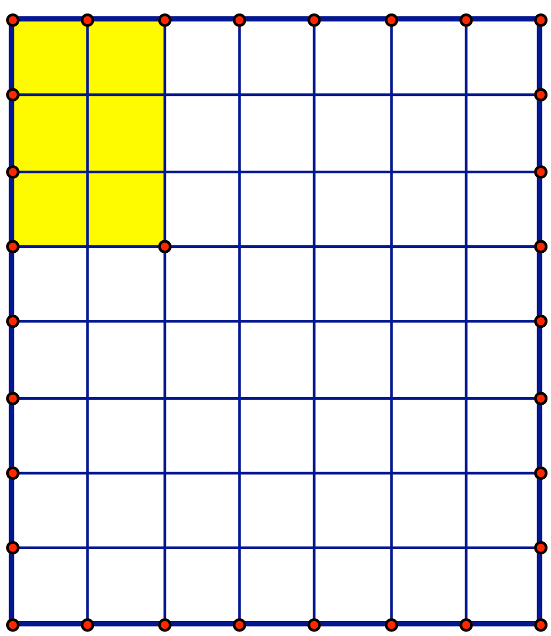
   iii. When is simplifying fractions important? When is it not important?
   A matter of opinion, but I tend to think it's important mostly for comparing answers with other people. Perhaps you will come up with some reasons of your own such as seeing relationships (which is easier to do with smaller numbers) and noticing patterns. I believe that patterns are more often obscured than exposed by simplifying fractions. Consider the example of the probabilities resulting from the sum of two dice, say getting a sum of 5: the denominator of 36 tells you the number of outcomes, and the 4 tells you the number of ways to get the sum. Simplifying to 1/9 obscures all of that useful counting information.

   iv. Why do the denominators need to be the same when adding (or subtracting)

fractions?
Draw a picture to show that the number of pieces might be added but since the pieces are of different size, there's no one denominator that will describe accurately what's going on.



v. Why do the denominators **not** need to be the same when multiplying two fractions? **Draw a picture** showing that multiplication of whole numbers involves perpendicular axes, making a rectangular region; in the same way, multiplying fractions results in drawing some perpendicular lines that automatically subdivide the pieces to a common size.



vi. Why is $\frac{2}{3}+\frac{5}{3}=\frac{7}{6}$ wrong and $\frac{2}{3}\times\frac{5}{3}=\frac{10}{9}$ right?

Units are an important reference here. When we add 2 inches plus 5 inches we get 7 inches, not 7 (inches + inches). The same thing works for thirds. On the other hand when we multiply 2 inches times 5 inches we get 10 (inches times inches).

vii. Why is dividing by 1/2 the same as multiplying by 2?
Models of division are really important. Two of the basic ones are (1) to divide objects into groups with a fixed number of groups: how many objects in each group? and (2) to divide objects into groups with a fixed number of objects in each group: how many groups? Either model works here: if we have only half a group, then we need twice as many objects for each whole group. If we have half an object per group, then the number of groups is twice the number of objects.

As a good additional exercise, or perhaps as a way of leading them toward the above explanation, ask your participants to explain a word problem or real-world situation in which you would want to compute 10 divided by 1/2. Can they distinguish situations that call for multiplying by 2?

Another additional exercise to focus on models of division as well as units is to compare the meaning of 10 dollars divided by 2 and 10 dollars divided by 2 dollars.

viii. How can common denominators help in dividing fractions?
Alternative algorithms are always important. Here, the idea of **units** also helps. For instance, dividing 3 feet by 4 inches is not so easy; you want a common unit first, 36 inches divided by 4 inches is 9. Similarly

$$\frac{2}{3}\div\frac{5}{7}=\frac{2\times 7}{3\times 7}\div\frac{5\times 3}{7\times 3}=\frac{14}{21}\div\frac{15}{21}$$

is a great way of showing why the result of this division is 14/15.

2. *Smallest Sum* (From Sam Vandervelde)
Using the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try 99/96 + 97/98 but that is not the smallest sum.

Easier problems are always a good idea. You can solve this problem with the numbers 1, 2, 3, and 4 rather than these big numbers and you'll find it a lot easier to think about. Then you need to check that the same pattern still works with the bigger numbers! You can also pretty quickly narrow it down to two possibilities with the smaller numbers on top and the bigger ones in the denominator, and then compare them by **subtracting**. Here it would be a bad idea to make a common denominator and simplify; instead it's much better to leave things as separate fractions with 98 and 99 in the denominator, and compare more directly. You'll definitely use the fact that smaller denominators make bigger fractions!

3. *Simpsons* (with thanks to http://www.cut-the-knot.com)
   i. Lisa's lemonade stand sells 20 cups of lemonade for 30 cents each and 80 cookies for 50 cents each. What is the average price per item?
   ii. Bart's lemonade stand sells 80 cups of lemonade for 40 cents each and 20 cookies for 60 cents each. What is the average price per item?
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   This is an illustration of "Simpson's Paradox", which states that **group averages** can differ in the opposite way from subgroup averages. Here we have Lisa being cheaper for both lemonade and cookies, and yet more expensive on average because she sells more of the more expensive cookies and fewer of the cheap cups of lemonade. Although Bart's ad copy might mention that he has a lower average price than Lisa, that's clearly misleading.

4. *More Simpsons*
   i. In their first basketball practice, Bart makes 5 out of 11 free throws while Lisa makes 3 out of 7. Who is the better free throw shooter?
   ii. In their second basketball practice, Bart makes 6 out of 9 free throws while Lisa makes 9 out of 14. Who is the better free throw shooter?
   iii. Who is the better free throw shooter?
   This takes the previous problem and puts it in a setting where the **real-world** implications are much more ambiguous. Bart did better each day, but Lisa did better overall. The fraction made each day was pretty different, and Lisa got more attempts on the day where everyone was making more. Should we favor Lisa because she did better overall, and there's no reason to separate the two days? Or should we figure Bart did better each day, so regardless of the circumstances it's better to let Bart shoot?

5. *In the Space Between*
   i. Name a fraction between 11/15 and 7/10.
   Most people name the **average** of the two, which involves a lot of work with common denominators. You can also talk about whether 21.5/30 is a fraction or not. Some people take the faster and easier approach and compare 110/150 and 105/150 and point out you could choose any denominator between 105 and 110. Choosing 108 seems good since it simplifies a lot: 108/150 is 18/25.

   ii. Name the fraction with smallest denominator between 11/15 and 7/10.

Hm, 18/25 - that's the **mediant** of the two fractions, found by adding the numerators and adding the denominators, just like you've always told them they shouldn't do when adding fractions. The mediant has many interesting properties; in some circumstances (but not this one!) it gives the fraction in between with smallest denominator. Also, the mediant has the interesting behavior of being dependent on the representation of the fraction and not only its numerical value: the mediant of 2/4 and 11/15 is not the same as the mediant of 1/2 and 11/15.

To get the really smallest denominator, probably the most commonly successful strategy is to use an **alternate representation**, converting the fractions to decimals in order to find that 5/7 lands between these, and no fraction with smaller denominator is in there.

iii. (Adapted from davidbau.com, which in turn adapted it from Gelfand and Shen *Algebra*)
First you draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into 7+13 = 20 equal pieces. How many marks are on each piece?

I often give another problem or two about mediants before this problem, for example: A bicycling team has 7 water bottles to share among 10 people, and another one has 11 bottles to share among 15 people. If they all share the water, how much does each person get? This shows really clearly why 18/25, the mediant, is definitely in between 7/10 and 11/15.

Then for this problem, you can quickly see that the end pieces have no marks on them. There are 18 other pieces, and 18 other marks. Drawing a picture starts to suggest that there's one mark on each piece, but some marks are really close together - is it a miracle that a cut lands between each one? Of course not. There can't be two red marks or two green marks on any one piece since those marks are too far apart. There can't be one red and one green mark on the same piece, since the mediant shows you where to find a cut line that lands between those two marks.

Warm-up Problem 2

* Write **three** positive integers in a line.
* In the space just below and between each pair of adjacent integers, write their difference.
* Can you arrange it so that each integer 1 through 6 appears exactly once?
* Example: 6 2 3

  4 1

  3 doesn't work: two 3s, no 5.
* How about starting with **four**? What goal now?
* How many ways can this be done?

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August 26, 2010

Problem vs Exercise

* Learn the problem solving salute!

Fraction Concept Quiz

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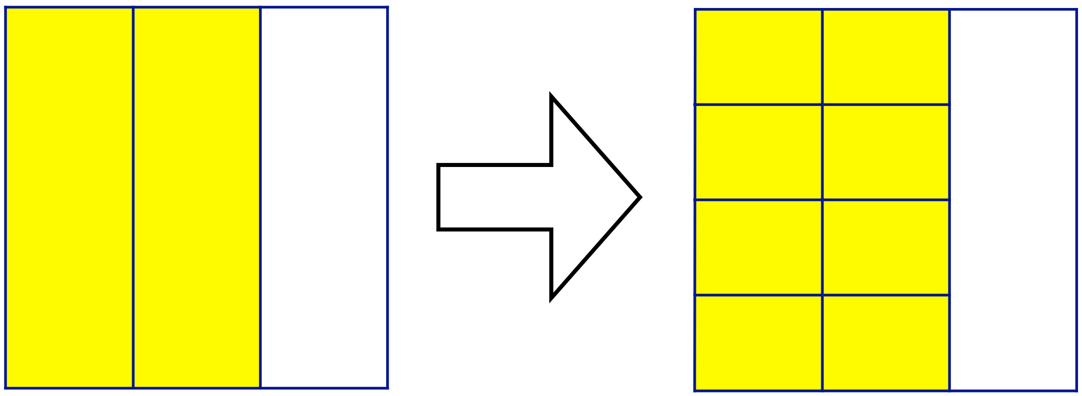

Fraction Concept Quiz

* How do you simplify a fraction?
  * Why is this often called “reducing”? In what ways is this a good name? A bad name?

Fraction Concept Quiz

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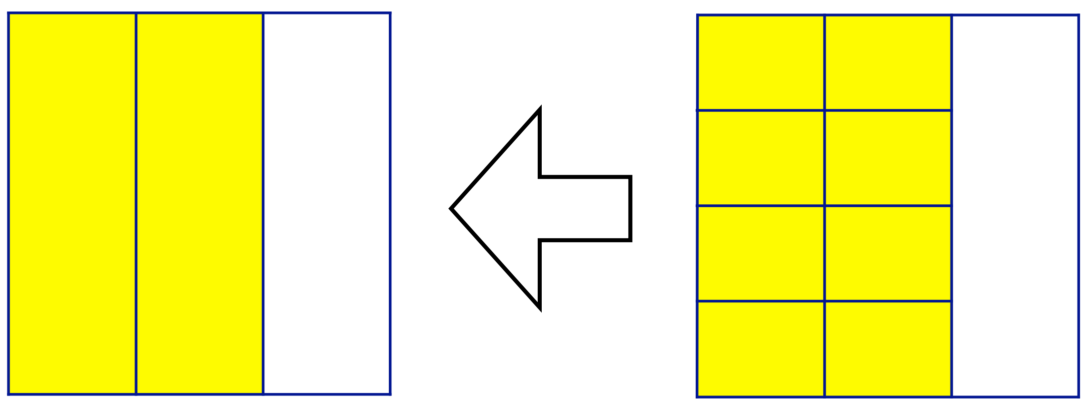
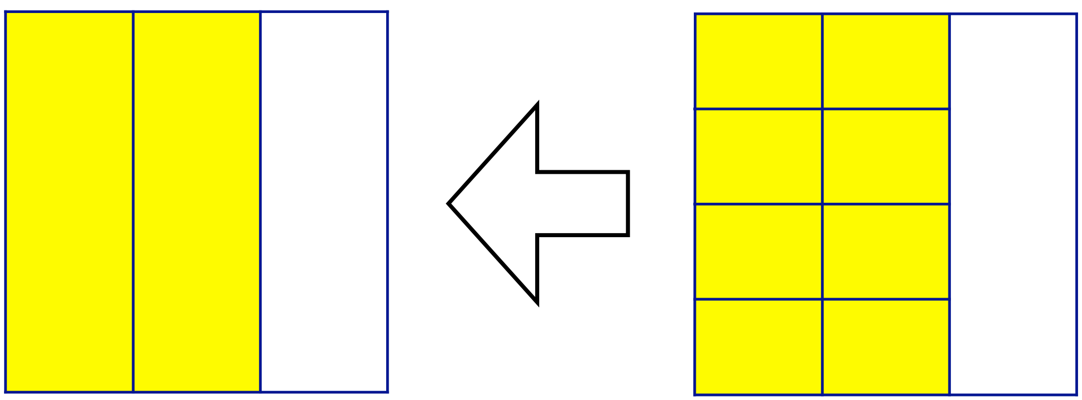


Fraction Concept Quiz

* We're reducing the numerator and denominator. We're reducing the number of pieces. But the value doesn't change, and the size of each piece increases!



Fraction Concept Quiz

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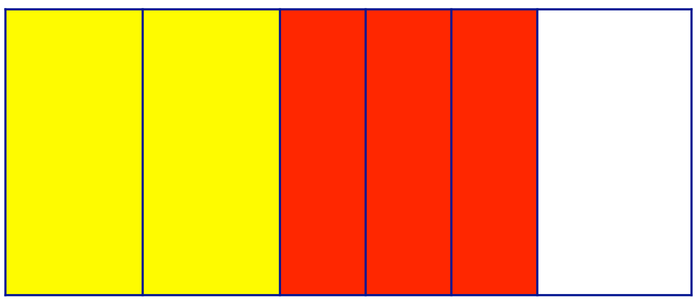
Fraction Concept Quiz

* Why are common denominators necessary for addition but not multiplication?
* 2 fifths plus 3 eighths is 5 ... whats?



Fraction Concept Quiz

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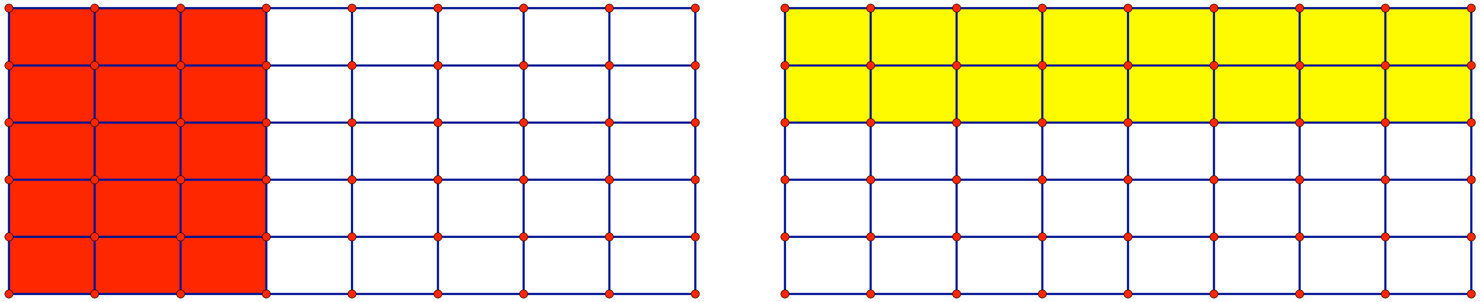
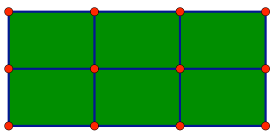


* Common denominators means to draw them perpendicular, not parallel.

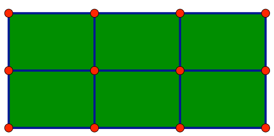
Fraction Concept Quiz

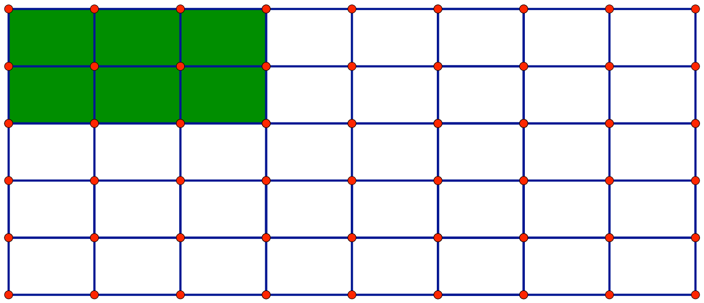
* Multiplication already has them perpendicular!
* 2 times 3 is 6,



Fraction Concept Quiz

- Multiplication already has them perpendicular!
- 2 times 3 is 6, and so 2/5 times 3/8 is 6/40





Think of the unit as the big rectangle, versus as the individual small ones! Multiplication ***is*** a change of unit.

Fraction Concept Quiz

* How can common denominators help you divide fractions?

$$\frac{2}{3} \div \frac{5}{7}$$

Fraction Concept Quiz

* How can common denominators help you divide fractions?

$$\frac{2}{3} \div \frac{5}{7} = \frac{2 \times 7}{3 \times 7} \div \frac{5 \times 3}{7 \times 3} = \frac{14}{21} \div \frac{15}{21}$$

Fraction Concept Quiz

- How can common denominators help you divide fractions?

$$\frac{2}{3} \div \frac{5}{7} = \frac{2 \times 7}{3 \times 7} \div \frac{5 \times 3}{7 \times 3} = \frac{14}{21} \div \frac{15}{21}$$

And now we are done?

- Twenty-firsts are a unit here!

A Fraction Story

* "It's hard to paint my living room," said the old man.
* "Maybe the job would be less scary if you only painted half each day." replied his friend.
* "That's a good idea."

A Fraction Story

* “So how did it go?” the friend asked a couple of weeks later.
* “Well, it seemed like a good idea. The first day was a bit of a job. The second day was easier. But after a few days it started getting really hard to paint only half!”

Smallest Sum

* Using the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try 99/96 + 97/98 but that is not the smallest sum.

Smallest Sum

- Big denominators are good.
- Then does it go 96/98 + 97/99, or is it 97/98 + 96/99?
- Don’t make common denominators and add; instead, notice that you are trading 1/98 for 1/99, and you’d rather have the 1/99 for the smallest sum.

Another Fraction Story

- The boss said “You’ve earned a vacation. Here’s $1000 so you can afford a week at that great hotel.”
- The employee replied “Unfortunately, there’s 40% tax on any money you give me as a bonus. So that would leave me owing $400.”
- “No problem! Here’s an extra $400.”

Another Fraction Story

* “Unfortunately, I have to pay 40% tax on that as well.”
* The boss muttered “Let’s see. That’s $1000, plus $400, plus $160, plus $64, plus ...”
* While the boss was punching away on the calculator, the secretary said - without a calculator! - that the total was about $1666.66. How?

Another Fraction Story

* The secretary figured that if the boss paid $\$x$, the employee would pay 40% of $\$x$ in taxes, and therefore keep 60% of $\$x$. So 60% of $\$x$ is equal to $1000, and thus $x = 1000 / 0.6$, or about $1666.66.
* By the way, what did you assume about the people in the story? How does that affect your interpretation of the story?

In the Space Between

* Name a fraction between 11/15 and 7/10.
* Common denominator? OK, 150, so we have 110/150 and 105/150, so that gives us some choices with denominator 150.
* Least common denominator? OK, 30, so we have 22/30 and 21/30, so that makes 21.5/30 then.
* Integer fraction? OK, 43/60 then.

In the Space Between

* Name the fraction with smallest denominator between 11/15 and 7/10.
* Let's "add" the fractions to get 18/25.
* Oh, you don't like that kind of addition? What does it mean? Imagine that you made 11 of 15 free throws one day and 7 of 10 the next day. What happened overall?

In the Space Between

- Name the fraction with smallest denominator between 11/15 and 7/10.
- OK, so 18/25 is between, but is it the best?
- Alternate representation: think of decimals! We want to be between 0.73333 and 0.7.
- Maybe you know that $5/7 = 0.714285\ldots$

In the Space Between

* First you draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into 7+13 = 20 equal pieces. How many marks are on each piece?

In the Space Between

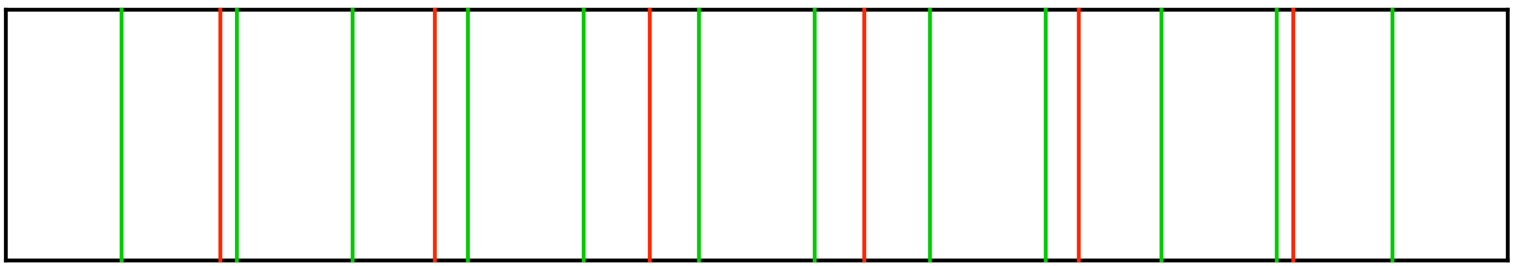
* How many marks are there?
* Right, for the 7ths you get 6 marks, and for the 13ths you get 12 marks, 6+12=18.
* The end pieces have no marks: 1/20 doesn't reach to the first 1/7 or 1/13 mark.
* But some of the marks in the middle are really close together!

In the Space Between

* Could a piece that is 1/20 long have two red marks that are 1/7 apart?
* Could a piece that is 1/20 long have two green marks that are 1/13 apart?
* Could a piece that is 1/20 long have one red and one green mark?

In the Space Between

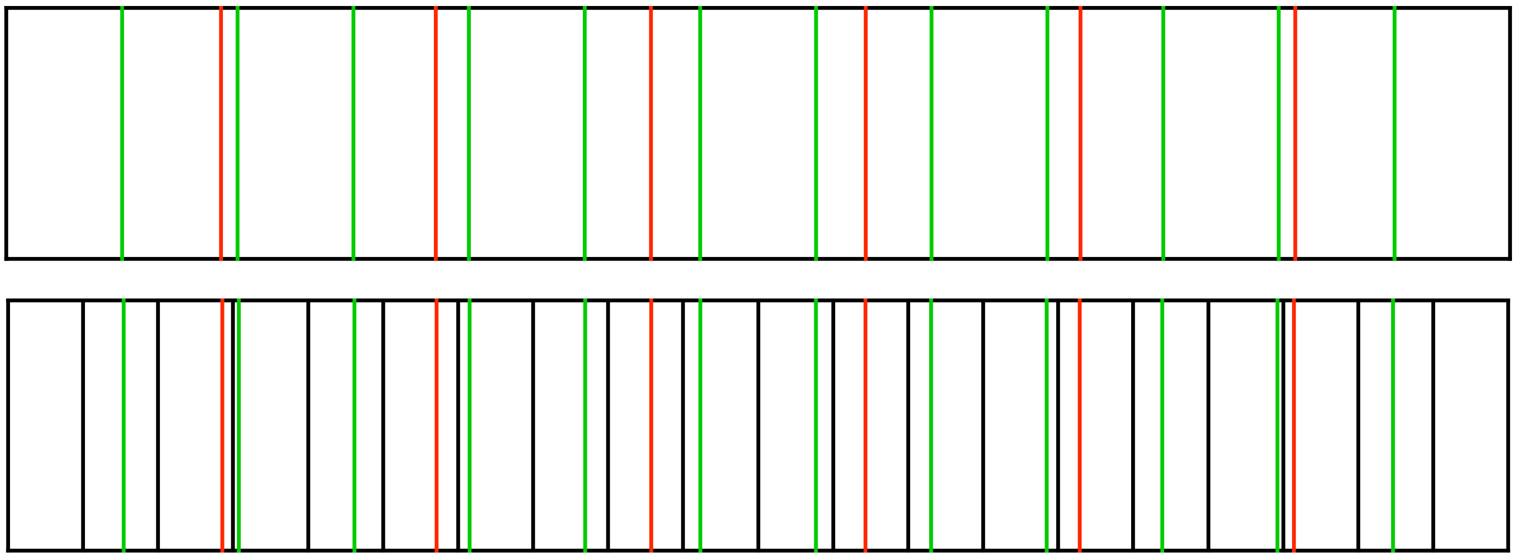
- Could a piece that is 1/20 long have one red and one green mark?



- Gosh, they are close together.

In the Space Between

- Could a piece that is 1/20 long have one red and one green mark?



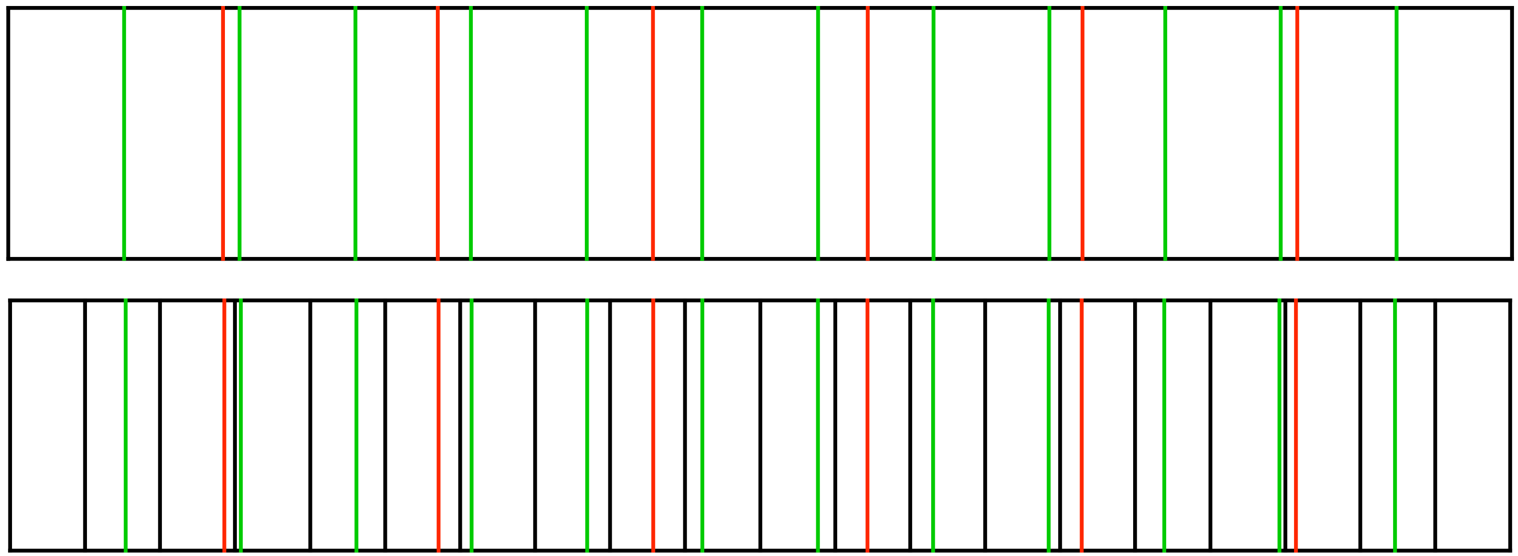
- The black lines magically land between!?

In the Space Between

* Seven students share one extra-large pizza. Thirteen students share two extra-large pizzas.
* What happens when they all decide to share?
* Twenty students share three extra-large pizzas.
* Thus, 3/20 is between 1/7 and 2/13.
* Or you could use the free-throw example, like finding 18/25 between 7/10 and 11/15.

In the Space Between

- Could a piece that is 1/20 long have one red and one green mark?



- The black lines have to land between!

Puzzle Bonus

- Where is large bigger than extra large?
- How can seven be half of twelve?
- Make 101 – 102 = 1 true by moving only one digit.
- Answers: Rome (L = 50, XL = 40). Again Rome: Take the top half of XII. And move the 2 up a bit.

Summary

- Think about **concepts**, not only algorithms.
- Pay attention to what is a **problem** and what is only an exercise.
- Learn techniques, tools, and strategies to turn problems into exercises, so you can try harder problems!