Favorite problems from the UWM Math Circle

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UWM Math Circle

- Started in September 2011
- Three faculty members from UWM Math Department

(Boris Okun, Chris Hruska, Gabriella Pinter)

- Students in grades 7-12
- Small circle – open ended problem solving
- Weekly meetings

A few memorable problems

- Pile splitting with tootsie rolls
- Cutting up a pentagon
- Superfactorials
- Hunt for isosceles triangles
- Cookie jar

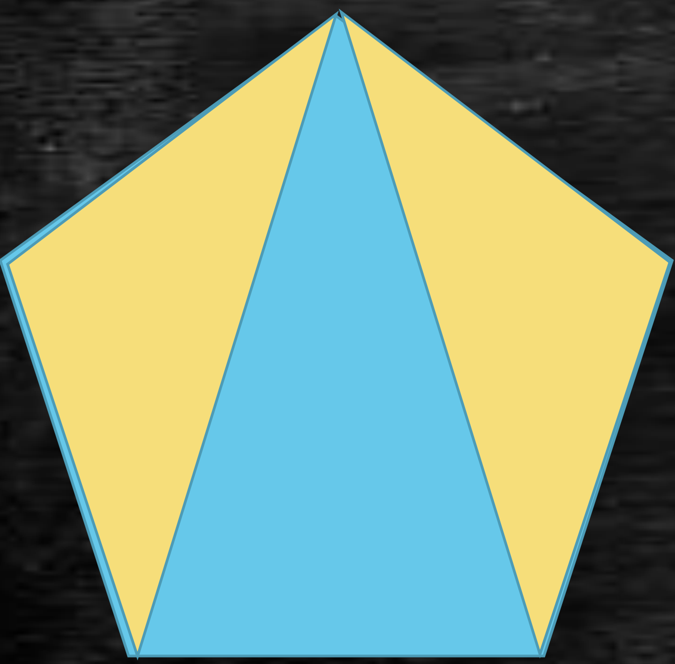
Pile splitting with tootsie rolls

Start with several piles of chips. Two players move alternately. A move consists of splitting every pile with more than one chip into two piles The player who can make the last move wins. Is there a winning strategy for the first or second player?

Memorable circle – students got absorbed in playing the game with tootsie-rolls

•started with one pile
•developed winning and losing numbers for the first (starting) player
•lots of question – significance of 2^k-1 started to emerge

Cutting up a pentagon

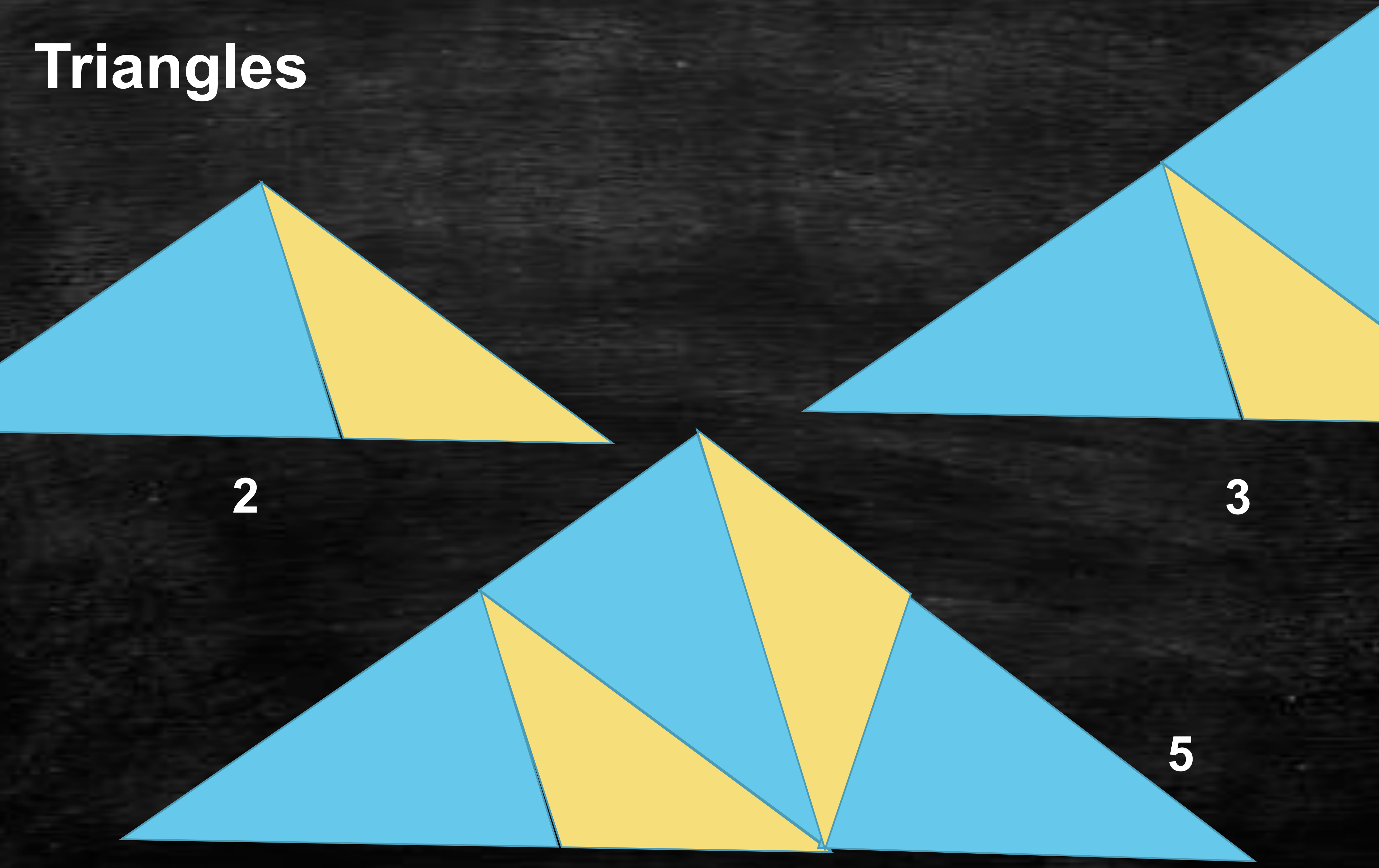


Cut several regular pentagons of th
same size into three triangles as
shown in the left. Make a triangle b
putting together

- two triangles
- three triangles
- five triangles

Can you continue? How? Why?

Triangles



Superfactorials

Cross out factors of the form n! in the product (1!)(2!)(3!) … (100!) so that the result is a square of an integer.
$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

What's the smallest number of factors you need to cross out to achieve this?

IDEAS:

1! is a square

$99! \cdot 100!$ is a square

$(n^2 - 1)! \cdot (n^2)!$ is always a square – so we are down to crossing out 81 factors

Is the original number a square? – NO

Difficulties with powers

$M = 1! \cdot 2! \cdot \ldots \quad 100! = 1^{100} \cdot 2^{99} \cdot 3^{98} \cdot 4^{97} \cdot \ldots \cdot 97^{4} \cdot 98^{3} \cdot 99^{2} \cdot 100^{1}$

M is not a square since 47^{even} 94^{odd} , and no other factors above contain a factor of 47

IDEA → consider primes and their exponents

only need primes < 50 but still a long process - 50!

A week later:

$M = 1^{100} \cdot 2^{99} \cdot 3^{98} \cdot 4^{97} \cdot \ldots \cdot 97^{4} \cdot 98^{3} \cdot 99^{2} \cdot 100^{1}$

$= 1^{100} \cdot 3^{98} \cdot \ldots \cdot 97^{4} \cdot 99^{2} \cdot 2^{99} \cdot 4^{97} \cdot \ldots \cdot 98^{3} \cdot 100^{1}$

$= K^{2} \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 96 \cdot 98 \cdot 100 = K^{2} \cdot 2^{50} \cdot 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 49 \cdot 50$

ow to go on?

$$M=1! \cdot 2! \cdot \ldots \quad n!$$

ame idea will work for any n=4k, where k is a
sitive integer
What if n=4k+1?
What is n= 4k+2 or 4k+3 ?

o is M ever a square? Is there always a prime
umber between k and 2k? (Bertrand’s postulate –
dős’ proof)

et $n!! = 1! \cdot 2! \cdot \ldots \quad n!$, and consider super-super
ctorials:

$n!!!= (1!!)\ (2!!)\ (3!!)\ \ldots\ (n!!)$ – will this ever be a
uare?

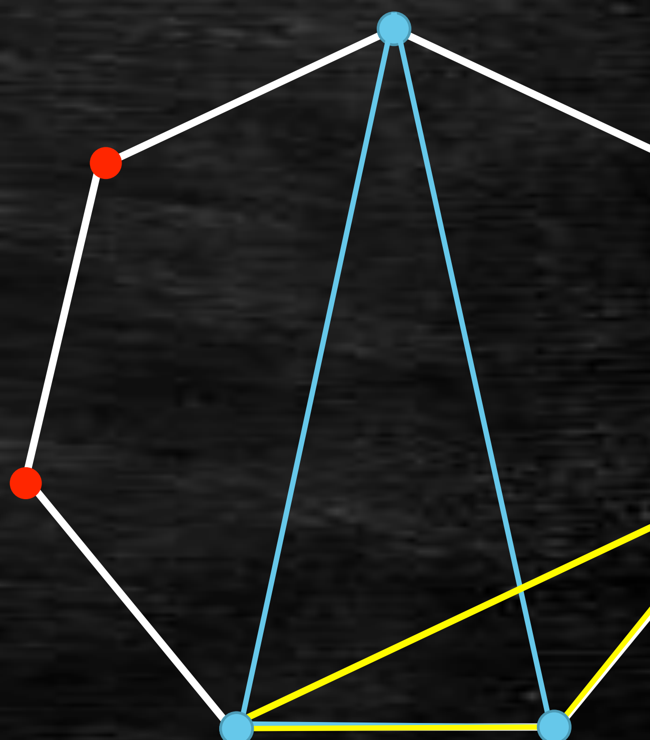
The hunt for isosceles triangles

The vertices of a regular heptagon are colored red and blue. Can we always find vertices of the same color which form an isosceles triangle?

IDEAS: adjacent vertices – same color

consider their neighbors

symmetry



Many new questions and ideas

ow many isosceles triangles are there?
onsider the jumps from one color to the same color
wise – when does an isosceles triangle exist?
rcular' partitioning
umber the vertices, and put them in two sets
rding to their color – arithmetic sequence

eneralization: consider other regular polygons - An
eles triangle exists if the regular polygon has n=5 ,
n> 8 vertices.
f: different cases for n odd or even – explicit
truction

hat if we have more colors?

Operation Cookie Jar

There are 15 cookie jars, numbered consecutively from 1 to 15. The number of cookies in each jar is equal to the number of the jar. A “move” consists of choosing one or more jars, then removing one or more cookies from the chosen jars - but the same number of cookies from each jar. How can we empty all the jars? How many moves?

THANK YOU !

List of problems with hints and questions can be picked up after the talk.

or

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