Polynomials and Pyramids. GEM Seminar

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Figure 1: How many golf balls are there?

1 In dimension two

Consider first the 2-dimensional version, the triangular numbers. The number of discs in a triangular array is obtained by summing up along the rows:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} := \binom{n+1}{2}$$

(these numbers are called binomial coefficients).

The number $\binom{k}{2}$ has a combinatorial interpretation: it's the number of ways to choose 2 objects out of k identical ones. Why? Because the first can be chosen in k ways, and the second in $k-1$ remaining ways. But we counted the same pair twice (2 is the number of ways to order two objects), so we divide by 2.

Likewise, the number of ways to choose m objects out of k is

$$\binom{k}{m} = \frac{k(k-1)\dots(k-m+1)}{m!}.$$

If our objects are the numbers $1, 2, \dots, n$ then a choice of two is a choice of i and j such that

$$1 \leq i < j \leq n.$$

Likewise for more than two objects.

What about choices with repetition? For two objects, this amounts to counting all pairs i, j such that

$$1 \leq i \leq j \leq n.$$

The problem reduces to the previous one if we add 1 to j. Then

$$1 \leq i < (j+1) \leq n+1,$$

and the number of such choices is $\binom{n+1}{2}$. Likewise, the number of ways to choose 3 objects, with repetitions, out of n is the number of triples i, j, k such that

$$1 \leq i \leq j \leq k \leq n,$$

and hence the number of solutions to

$$1 \leq i < (j+1) < (k+2) \leq n+2,$$

that is, $\binom{n+2}{3}$. And so on.

But is this a coincidence that the number of discs has a combinatorial meaning? Of course, not! Each disc in the 2-dimensional pyramid has two coordinates: the number of its row (counting from the above), and its position in the row (from the left). Call these numbers j and i. Then $1 \leq i \leq j \leq n$, so the total number is $\binom{n+1}{2}$, as we already know.

2 In dimension three

We can use both approaches that worked in dim 2. On the one hand, each ball in the pyramid has three coordinates: the number of its "floor" (counting from above), and the two coordinates on the triangle on that floor, as in dim 2. This gives us three numbers, k, j, i such that

$$1 \leq i \leq j \leq k \leq n,$$

and the number of solutions is $\binom{n+2}{3}$. These are the pyramidal numbers.

On the other hand, we can find the number of balls by summing the number of balls on each floor:

$$\sum_{k=1}^{n} \binom{k+1}{2} = 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2}.$$

We conclude that

$$1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}. \tag{1}$$

Let us double-check. The sum $1 + 3 + 6 + \cdots + \frac{n(n+1)}{2}$ would be easy to compute if we knew that the general term, $\frac{k(k+1)}{2}$ was $F(k) - F(k-1)$ for some function F. Then the sum would "telescope":

$$\sum_{k=1}^{n} \binom{k+1}{2} = (F(1)-F(0))+(F(2)-F(1))+\cdots+(F(n)-F(n-1)) = F(n)-F(0).$$

So we need to check that

$$\frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6} - \frac{(k-1)k(k+1)}{6},$$

which is indeed true.

In a similar way, one obtains a more general identity:

$$\sum_{k=1}^{n} \binom{k+m-1}{m} = \binom{n+m}{m+1}$$

which follows from the identity

$$\binom{n+m-1}{m} = \binom{n+m}{m+1} - \binom{n+m-1}{m+1}. \tag{2}$$

3 Integer-valued polynomials

Let's consider the problem: *which polynomials $f(x)$ have integer values for all integer values of the argument x?*

Of course, a polynomial with integer coefficients will do. But there is more: $f(x) = \binom{x}{k}$ is such a polynomial (of degree k) because it has a combinatorial meaning.

Theorem. Every polynomial of degree n with the desired property have the form

$$f_n(x) = \sum_{k=0}^{n} c_k \binom{x}{k}$$

where all the coefficients c_k are integers.

Proof. Induction on n. For $n = 0$, this is trivial.

Let $f_{n+1}(x)$ be such a polynomial. We can write

$$f_{n+1}(x) = \sum_{k=0}^{n+1} c_k \binom{x}{k},$$

but we don't know yet that all c_k are integers.

We know that $f_{n+1}(x)$ is an integer for all integer values of x, so $f_{n+1}(x+1) - f_{n+1}(x)$ is also an integer. But

$$f_{n+1}(x+1) - f_{n+1}(x) = \sum_{k=0}^{n+1} c_k \left(\binom{x+1}{k} - \binom{x}{k} \right) = \sum_{k=1}^{n+1} c_k \binom{x}{k-1},$$

the last equality follows from (2). Now, the RHS has degree n, hence, by the induction assumption, all $c_k,\ k = 1, \dots, n+1$ are integers.

It remains to see that c_0 is also an integer. Indeed,

$$c_0 = f_{n+1}(x) - \sum_{k=1}^{n+1} c_k \binom{x}{k},$$

and the RHS takes integer values when x is an integer. Hence c_0 is an integer.

4 Square-based pyramid

The number of balls is $1 + 4 + 9 + \cdots + n^2$. One way to find this sum is to use (1). Indeed,

$$k^2 = 2\binom{k+1}{2} - k,$$

hence

$$\sum_{k=1}^{n} k^2 = 2\sum_{k=1}^{n} \binom{k+1}{2} - \sum_{k=1}^{n} k = 2\binom{n+2}{3} - \binom{n+1}{2} =$$

$$\frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

Similarly one can find other sums of this kind:

$$1^m + 2^m + \cdots + n^m.$$

5 So, how many balls?

The number of golf balls was about 500. The closest pyramidal number less than 500 is

$$\binom{15}{3} = 455,$$

hence the side of the pyramid in Figure 1 is 13.

6 Exercises

1. Find a closed formula for the pentagonal numbers.
2. Find $1^3 + 2^3 + 3^3 + \ldots n^3$.
3. Let $f(x)$ be a polynomial of degree n. Prove that $f(x+1) - f(x)$ has degree $n-1$.
4. The numbers $1, -1, 5, 25, 65, 131, \ldots$ are the values $f(0), f(1), f(2), \ldots$ for some polynomial $f(x)$. Find the least possible degree of $f(x)$.
5. Can the numbers $1, 2, 4, 8, 16, 32, \ldots$ be the values $f(0), f(1), f(2), \ldots$ for some polynomial $f(x)$?
6. Let $f(x)$ be a polynomial of degree n. Find the degree of the polynomial $g(x) = f(x+1) - 2f(x) + f(x-1)$.