Circle on the Road 2013. Julia Robinson Festival

Lesson on Word Arithmetic (Encrypted Puzzles)

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In these puzzles, we will be working with numbers that have been encrypted: the digits in these numbers have been replaced by letters. Our goal will be to decipher these puzzles.

Verbal arithmetic problems are both educational and entertaining. There is a lot of detective work involved in solving them. At the same time, working on these puzzle-like problems give students a good chance to deepen their understanding of numbers and arithmetic operation.

The target audience of this presentation are grade 4-6 students.

0.1. Math Warm-up

Warm-up 1. Six glasses are placed in a row. The first three are filled with water, and the last three are empty. Your goal is to position empty and filled glasses in alternating order. The difficulty is that you are allowed to move only one glass out of the six. How can you complete this task?

Warm-up 2. Can you throw a ball in such a way that after some time it would stop moving, change its course and start moving in the opposite direction?

0.2. Encrypted Problems

Problem 1. Suppose somebody shows you the addition problem displayed in Figure 1.

$$\begin{array}{r} \text{ME} \\ +\quad \text{M} \\ \hline \text{ASA} \end{array}$$

FIGURE 1.

This problem looks strange since the numbers are not quite numbers. Instead, they have been transcribed using letters. The truth is that the original problem was encrypted: all the digits were replaced by the letters. Each letter denotes a digit: the same letters stand for the same digits, and different letters stand for different digits. Our goal is to reconstruct the original numbers.

How should we start? When a cryptanalytic (the person whose job is to decrypt secret messages) starts working with a ciphered piece of text, she looks for a weak spot: a symbol that is the easiest to guess. We will do the same.

Let us notice that none of the addends in the hundreds column have any value in the hundreds place. Therefore, A is equal to the carry from the tens column. We are adding up two numbers – for that reason, a carry cannot be greater than 1. Hence, A is equals to 1 (see figure 2.)

Another possible approach for deducing the value of A would be to use estimates: ASA is the sum of a two-digit number and a one-digit number. Since ASA cannot be greater than 200, its hundreds digit A should be equal to 1.

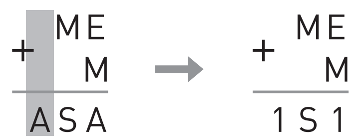

Figure 2.

To make a guess about the value of the next letter, we should concentrate on the addition in the tens column. The only number in this column is M, and nothing is added to it. Yet, the tens digit of the sum is different from M: it is equal to S. This difference can be explained only by the presence of a carry from the ones column. This carry (1) is added to M: the resulting sum yields a carry that gives us the hundreds digit of the sum. If M were less than 9, then $M+1$ would never produce a carry. Therefore, $M=9$ and $S=0$. The fact that $E=2$ is a freebie since we know the values of the rest of the digits (see figure 3).

The same result could have been deduced from the observation that a two-digit number added to a one-digit number will never be greater than $99+9=108$. Therefore, S has to be equal to 0. In a similar fashion, M has to be 9 because otherwise $ME+M$ would never be greater than 100.

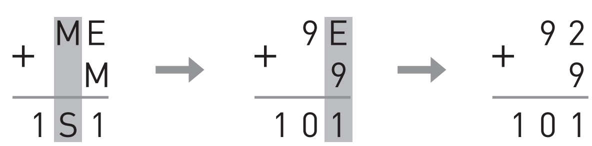

Figure 3.

For Teachers: Since this problem is not that difficult, some students may guess the answer right after you write the problem on the board. However, guessing is not the right approach for problems of this kind. A solution should always include arguments that justify that there is no other way to decrypt this problem.

A couple more easy problems can be presented if needed: AA+AB=CBA, ELF + ELF = FOOL, DAD+DAD=PAPA.

Comment 1: each of these problems should be written on the board as column addition.

Comment 2: The source of the ELF problem is “Sideway Arithmetic from Wayside School” by Louis Sachar.

Problem 2. Decrypt the problem in Figure 4. (The same letters stand for the same digits, and different letters stand for different digits.)

$$\begin{array}{r} \mathsf{S\,U\,P} \\ +\ \mathsf{S\,P\,U} \\ \hline \mathsf{U\,P\,S} \end{array}$$

FIGURE 4.

Problem 2 Discussion. This problem happens to be way more difficult for students than the previous one.

The addition in the hundreds column provides us with information that S is not greater than 4. Unfortunately, this fact seems to be of little use.

However, the addition in the tens column looks more promising. The numbers U, P, and a possible carry from the ones column are all added together there. This addition creates a number that ends with digit P. To make such an addition possible, the sum of U and the potential carry should be equal to either 0 or 10. Therefore, only two options exist: either there is no carry and $U = 0$, or there is a carry (which equals to 1) and $U = 9$. However, accepting the first option ($U = 0$) generates a conflict with the addition in the hundreds column: S definitely has to be greater than 0. Therefore, $U = 9$ (see figure 5).

$$\begin{array}{r} \mathsf{S\,U\,P} \\ +\ \mathsf{S\,P\,U} \\ \hline \mathsf{U\,P\,S} \end{array} \longrightarrow \begin{array}{r} \mathsf{S\,9\,P} \\ +\ \mathsf{S\,P\,9} \\ \hline \mathsf{9\,P\,S} \end{array}$$

FIGURE 5.

After U has been decrypted, it is easy to deduce from the addition in the hundreds column that $S = 4$. The value of letter P comes from the addition in the ones column: the sum of P and 9 ends with 4. Hence, $P = 5$ (see figure 6).

$$\begin{array}{r} \mathsf{S\,U\,P} \\ +\ \mathsf{S\,P\,U} \\ \hline \mathsf{U\,P\,S} \end{array} \longrightarrow \begin{array}{r} \mathsf{S\,9\,P} \\ +\ \mathsf{S\,P\,9} \\ \hline \mathsf{9\,P\,S} \end{array} \longrightarrow \begin{array}{r} \mathsf{4\,9\,P} \\ +\ \mathsf{4\,P\,9} \\ \hline \mathsf{9\,P\,4} \end{array} \longrightarrow \begin{array}{r} \mathsf{4\,9\,5} \\ +\ \mathsf{4\,5\,9} \\ \hline \mathsf{9\,5\,4} \end{array}$$

FIGURE 6.

Therefore, to our great surprise, we were able to restore the whole problem without having any information about the original numbers.

Problem 3. Decode: $AT+AT+AT=BAT$. (The same letters stand for the same digits, and different letters stand for different digits.)

Comment: If this problem is too difficult for the audience, we'll skip it.

Problem 3 Discussion. If you invite your students to take a crack at this problem, they would probably start by rewriting the equation in the column addition format. Then, they would try to break the encryption using the techniques from the previous problem. However, they will fail. (Try it, it is quite difficult to solve this problem that way.)

This problem needs a completely different approach. Some meditation on the problem brings out the fact that the sum on the right is composed of B hundreds and of AT ($B\times 100+AT$). Therefore, the equality can be simplified as: $AT+AT=B\times 100$. Since AT is a two-digit number, the sum on the left is less than 200. Therefore, $B=1$. Consequently, $A=5$ and $T=0$.

Problem 4. Calculate: $(H\times E\times R\times M\times I\times O\times N\times Y)/(W\times A\times N\times D)$. (The same letters stand for the same digits, and different letters stand for different digits.)

Problem 4 Discussion. Note that exactly 10 letters are used in this problem. Since there are 10 digits, then one of the letters stand for 0. Since this is a fraction, then the digit 0 should be on top. Therefore, this fraction is equal to 0.

Problem 5. Explain why these puzzles cannot be solved. (The same letters stand for the same digits, and different letters stand for different digits.)

a)
$$\begin{array}{r} KATHRIN \\ +\quad BELLA \\ \hline FRIENDS \end{array}$$

b)
$$\begin{array}{r} BAT \\ +\ RAT \\ \hline CAT \end{array}$$

The primary source for this lesson is the "Mathematical Circle Diaries, Year1" book, chapter 20.