Numerical Puzzles: Runaway Digits. (Lesson for grade 4-6 students. By Anna Burago, Prime Factor Math Circle.)

The lesson is devoted an exciting topic – numerical puzzles. Problems of this type are a valuable resource since they possess both an entertaining and an educational value. Students enjoy numerical puzzles because they treat them as brain-teasers and riddles. Teachers also appreciate these problems for several reasons. They help the students develop critical thinking and logical skills, as well as deepen children's understanding of the nature of numbers and of arithmetic operations.

Moreover, these puzzles carry an additional appeal to the students: with some persistence, every student is usually able to find the solution. The path to the solution will be defined by the individual balance of trial-and-error and logical analysis.

Easy Problem 1.

Little Max wrote an equality on the board. Little Bella erased three digits and replaced them with asterisks symbols. Restore the original equality.

$* + * = *8$

For Teachers: you may need to remind your students the meaning of the term "digit": a digit is a numeral from 0 to 9. Digits are used to form numbers in the same way as letters are used to form words.

Problem 1 Discussion.

The two one-digit numbers on the left add up to a two-digit number on the right. Let us try to deduce some information about the sum on the right. Each of the numbers on the left is less than 10. Therefore, their sum should be less than 20. Since this sum ends with an 8, it has to be equal to 18.

Now that we know the sum, let us concentrate on the addends on the left-hand side. If at least one of them were less than 9, then their sum would be less than 18. Therefore, we can unquestionably restore both left-hand side numbers to 9.

For Teachers: The students will definitely tell you that it is much faster to guess the solution to this problem than to deduce it. However, there are several objections to guessing. Firstly, a deduced solution serves as a proof that no other answers to the same problem are possible. Secondly, this particular solution illustrates the use of the estimations technique in numerical puzzle problems.

This easy problem introduces us to the type of problems we'll be working on today: numerical puzzles.

Problem 2.

$* * 5 - ** = 8$

Little Bella wrote an equality on the board. Little Max erased five digits and replaced them with asterisks. Restore the original equality.

Problem 2 Discussion.

Let us concentrate on the magnitude of the two unknown numbers. A two-digit number is subtracted from a three-digit number, and the result is less than 10. How big could this three-digit number be? Let us start with a rough estimate. For example, could this number be 200 or bigger? Definitely not. Otherwise, the number on the right would not be so small. Therefore, the first digit of the three-digit number is 1. To figure out the second digit of the number $1 * 5$, let us run one more estimate. This new estimate will be more precise. We start by rewriting this equality as follows: $1 * 5 = * * + 8$.

Notice that on the right we have a sum of a two-digit number and number 8. Even if we use the biggest possible two-digit number – 99, this sum will never be bigger than 107. Therefore, the center digit of the number $1 * 5$ is equal to 0. Thus, the only possible answer to this problem is: $105 - 97 = 8$.

For Teachers: Note that using the second (more precise) estimate alone is sufficient for solving this problem. However, the first (rough) estimate is such a handy and easy technique that it would be a shame not to demonstrate it here.

Problem 3.

Restore the missing digits (see Figure 1).

$$\begin{array}{r} * \, 9 \, * \\ + \quad * \, 7 \\ \hline * \, * \, 4 \, 3 \end{array}$$

FIGURE 1.

Problem 3 Discussion.

This problem is pretty straightforward. The ones digit of the top number is unquestionably restored to 6 using the column addition properties. This addition in the ones column carries 1 to the tens column. Therefore, the unknown digit in the tens column has to be equal to 4.

Two approaches are possible for restoring the hundreds digit of the top number.

Firstly, this digit can be figured out by using the properties of the column addition. Let us observe that the addition of this digit and number 1 (the carry from the tens column) results in a carry to the next (thousands) column. However, such a carry is possible only if this unknown digit is equal to 9 (since the sum of 1 and any other one-digit number is less than 10).

Secondly, we can figure out the same digit using estimates: if the sum of a three-digit number and a two-digit number is greater than a thousand, then this three-digit number should be greater than 900. Therefore, the hundreds-digit of the top number is 9.

After we figure out all the digits of the first and the second numbers, the third number is very easy to reconstruct. The answer is $996 + 17 = 1043$.

Problem 4.

Restore the missing digits (see Figure 2).

$$
\begin{array}{r}
 * * 7\,1 \\
- \quad * 9 * \\
\hline
 * 3
\end{array}
$$

FIGURE 2.

Problem 4 Discussion:

The estimates technique will prove to be useful for this problem as well. A number that is less than a thousand is subtracted from a number that is greater than a thousand, and the difference is less than a hundred. How big could this greater-than-a-thousand number be? It cannot be greater than 1098: even the biggest three-digit number (999) together with the biggest two-digit number (99) add up to 1098 only. Therefore, the top number is equal to 1071. At this point, the rest of the numbers are easily restored.

For Teachers: Most of the puzzles that we study today allow for two types of solutions. The first type uses the properties of the column addition algorithm. The second one relies on the estimates and the understanding of the quantitative relations between numbers. While children are usually well-trained to use the column addition algorithm, they are not accustomed to thinking about the ideas behind it. As a result, students tend to present the solutions of the first kind. Therefore, it is a good idea for a teacher to emphasize the estimates techniques, as they train children to think about the *meaning* of arithmetic operations.

Problem 5.

Restore the missing digits (see Figure 3).

```
    * *
+   5 2
  -----
    * 4
  * *
  -----
  * 8 *
```

FIGURE 3.

Problem 5 Discussion.

This problem is not as simple to figure out as the previous ones. The only quick observation is that the ones digit in the bottom number (∗8∗) has to be equal to 4.

From this point on, it is necessary to engage in logical analysis. We should explore the problem in order to make the strategic observation that will lead us to a breakthrough.

Let us concentrate on the unknown two-digit number in line four: ∗∗. This number is the product of the unknown two-digit top number and number 5. How could this product be less than 100? If this two-digit number were 20 or greater, then a number that is five times bigger would be at least 100. Therefore, the top number should be less than or equal to 19.

This discovery leads us to the conclusion that the tens digit of the top number is 1. To guess the value of the ones digit of this top number, one should resort to an intelligent trial-and-error strategy. The product of this unknown digit and number 2 ends on 4. Therefore, this digit is equal to either 7 or 2. At this point, the most straightforward way to proceed is to plug 7 and 2 into the ones-place. The goal is to generate a three-digit product with digit 8 in the center. $12 \times 52 = 624$ and $17 \times 52 = 884$. Therefore, the final answer is $17 \times 52 = 884$.

Problem 6 is left for independent practice.

Problem 6. Restore the missing digits (see Figure 4).

a)
$$\begin{array}{r} * * \\ + \quad * \\ \hline * * 8 \end{array}$$

b)
$$\begin{array}{r} * 6 * \\ + \quad 2 * 5 \\ \hline 6 3 8 \end{array}$$

c)
$$\begin{array}{r} 8 3 \\ \times \quad * * \\ \hline * 3 \\ 1 * * \\ \hline * * * * \end{array}$$

FIGURE 4.

Extra Problems

Problem 1.

Replace all the stars with integer numbers ranging from 0 to 9 in such a way as to get the correct equality:

a) $1 * \times * 1 = 1 * 1$ b) $* * * 7 - * * * = 8$

Problem 3.

Replace all the asterisks with numbers from 0 to 9 in such a way as to get the correct equalities (see Figure 5).

a)
$$\begin{array}{r} * 9 3 \\ + \quad * * \\ \hline * * 5 1 \end{array}$$

b)
$$\begin{array}{r} 2 9 \\ \times \quad * * \\ \hline * 7 \\ * * \\ \hline 3 * * \end{array}$$

c)
$$\begin{array}{r} * * \\ \times \quad * 8 \\ \hline * * \\ * * \\ \hline 3 * 6 \end{array}$$

FIGURE 5.

Problem 5.

In Figure 6, replace all the stars with numbers from 0 to 9 in such a way as to get the correct equality.

$$\begin{array}{r} * * \\ \times \quad 8 * \\ \hline * * * \\ * * \\ \hline * * * * \end{array}$$

FIGURE 6.