Wrangle 2012 – Practice 3 – Problems and Solutions

1. By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself. A natural number greater than 1 will be called "nice" if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

Solution:

Let $p(n)$ denote the product of the distinct proper divisors of n. A number n is "nice" in one of two instances:

(a) It has exactly two distinct prime divisors p and q, so that $p(n) = pq = n$;

(b) It's a cube of a prime number, $n = p^3$, so that $p(n) = p \cdot p^2 = p^3 = n$.

Now we need to show that these above are the only two cases. Suppose that another "nice" number existed that does not fall into one of these two categories. Then we can either express it in the form $n = pqr$ (with p, q prime, and $r > 1$), or $n = p^e$ (with p prime and $e \neq 3$).

In the former case, p, q, pq, and rq are proper divisors of n, so $p(n) \geq p \cdot q \cdot pq \cdot pr = (pqr)^2 = n^2 > n$.

In the latter case, $p(n) = p \cdot p^2 \cdot p^3 \cdot \ldots \cdot p^{e-1} = p^{\frac{e(e-1)}{2}}$. Hence, if n is "nice", then $p^{\frac{e(e-1)}{2}} = p^e$, so $\frac{e(e-1)}{2} = e$ whence $e = 0$ or $e = 3$. But $e = 0$ does not work since then $n = 1$.

Thus, listing out the first ten numbers to fit this form,

$2 \cdot 3 = 6,\ 2^3 = 8,\ 2 \cdot 5 = 10,\ 2 \cdot 7 = 14,\ 3 \cdot 5 = 15,\ 3 \cdot 7 = 21,\ 2 \cdot 11 = 22,\ 2 \cdot 13 = 26,\ 3^3 = 27,\ 3 \cdot 11 = 33$.

Summing these yields 182.

2. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

Solution:

$$y^2 + 3x^2y^2 - 30x^2 = 517$$
$$y^2 + 3x^2(y^2 - 10) = 517$$
$$(y^2 - 10) + 3x^2(y^2 - 10) = 507$$
$$(y^2 - 10)(1 + 3x^2) = 507 = 3 \cdot 13^2$$

Since x and y are integers, and $1 + 3x^2 \in \{1, 3, 13, 169, 507\}$. Clearly, $1 + 3x^2$ is not a multiple of 3, so $1 + 3x^2 \neq 3\ or\ 507$. If $1 + 3x^2 = 1$, then $x = 0$, so $y^2 = 517$, which is impossible since 517 isn't a perfect square. If $1 + 3x^2 = 169$, then $x^2 = 56$, impossible. Hence $1 + 3x^2 = 13$, so $x^2 = 4$. Thus $y^2 - 10 = 39,\quad y^2 = 49$. Therefore, $3x^2y^2 = 3 \cdot 4 \cdot 49 = 588$.

3. Prove that every scalene triangle contains two sides such that the quotient of their lengths is less than or equal to 1 but greater than 3/5.

Solution: Suppose that there exists a scalene triangle that does not satisfy the conditions. Let a, b, c be the lengths of the sides, and suppose that $a < b < c$. Then $\frac{a}{b} \leq \frac{3}{5}$, and $\frac{b}{c} \leq \frac{3}{5}$, whence $a \leq \frac{3}{5}b$, and

$b \le \frac{3}{5}c$, so that $a \le \frac{3}{5}b \le \left(\frac{3}{5}\right)^2 c$. Hence $a + b \le \left(\frac{3}{5}\right)^2 c + \frac{3}{5}c = \frac{24}{25}c < c$, which is impossible by the triangle inequality.

4. Suppose we start with an $n \times n$ square consisting of n^2 white cells. We can pick any cell and then paint black this cell, all cells that share an edge with this cell, and all cells in the same column with the chosen cell. We can repeat this process any number of times; it's allowed to paint a cell black more than once. What is the least number of times this process should be repeated so that all n^2 cells become black?

Answer: n. (Prove it!)

5. Suppose that points A, B, C are vertices of a scalene triangle. How many points D in the plane of ΔABC have the property that quadrilateral $ABCD$ has at least one axis of symmetry?

Solution: If a quadrilateral has an axis of symmetry, then every vertex maps by the reflection across this axis onto a vertex of the quadrilateral. Thus every such axis is either the perpendicular bisector of a side of the triangle ABC, or contains an entire side. Thus if the triangle ABC is not a right triangle, then there are 6 points D satisfying the condition (construct them!). If ABC is a right triangle, there are only 3 such points (Why? Construct them!).

6. Find the number of solutions of the given equation for various values of the parameter b : $\sqrt{3x-5} = b - \sqrt{3x+11}$.

Solution: Let's rewrite the equation as $\sqrt{3x-5} + \sqrt{3x+11} = b$. Let $f(x) = \sqrt{3x-5} + \sqrt{3x+11}$. This function is defined on $[5/3, \infty)$, and it is increasing since both $\sqrt{3x-5}$ and $\sqrt{3x+11}$ are increasing. Thus if $b \ge f(5/3) = 4$, the equation has exactly one solution, and if $b < 4$, there are no solutions.

7. Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1,000,000. What is the integer nearest to S?

Solution: $1{,}000{,}000 = 2^6 5^6$, hence there are (6+1)(6+1) = 49 divisors, and 47 of them are proper divisors. All these 47 proper divisors with the exception of $\sqrt{1{,}000{,}000} = 10^3$ can be paired up so that the product of each pair is $pq = 10^6$. But then $\log(p) + \log(q) = \log(pq) = \log(10^6) = 6.$ Thus the total sum is $23 \cdot 6 + \log(10^3) = 141$.

8. Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\bar{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the area of A? ('92/10)

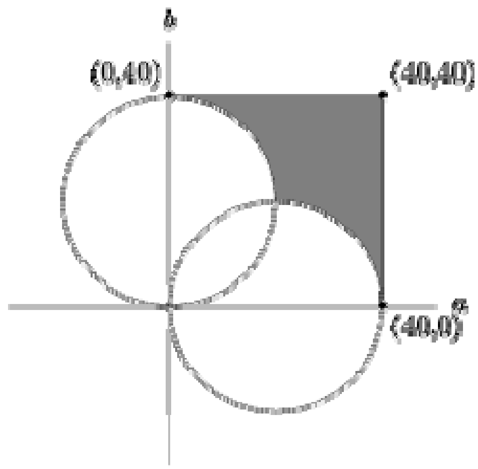
Solution:

Let $z = a + bi \implies \frac{z}{40} = \frac{a}{40} + \frac{b}{40}i$. Since $0 \leq \frac{a}{40}, \frac{b}{40} \leq 1$ we have the inequality

$0 \leq a, b \leq 40$ which is a square of side length 40.

Also, $\frac{40}{\overline{z}} = \frac{40}{a - bi} = \frac{40a}{a^2 + b^2} + \frac{40b}{a^2 + b^2}i$ so we have $0 \leq a, b \leq \frac{a^2 + b^2}{40}$, which leads to: $(a - 20)^2 + b^2 \geq 20^2$ $a^2 + (b - 20)^2 \geq 20^2$

We graph them:



We want the area outside the two circles but inside the square. Doing a little geometry, the area of the intersection of those three graphs is $40^2 - \frac{40^2}{4} - \frac{1}{2}\pi 20^2 \approx 571.68$