Math Wrangle MAA - Golden Section, February 25, 2012

1. In a triangle *ABC*, $\angle A = 120°$. Suppose that *D* is a point on the angle bisector of the angle *A*, and $AD = AB + AC$. Find the angles *CBD*, *BCD*, and *BDC*.

2. Suppose that the sum of the squares of two complex numbers *x* and *y* is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

3. Does there exist a trapezoid with the property that the (positive) difference of the lengths of its sides is bigger than the (positive) difference of the lengths of its bases?

4. Austin takes red and black cards out of a bag and arranges them on a table into two stacks. It is prohibited to place a card on top of a card of the same color. The 10^{th} and 11^{th} cards placed by Austin on the table are both red, while the 25^{th} card is black. What color is the 26^{th} card placed on the table?

5. What is the largest positive integer *n* for which there is a unique integer *k* such that $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$?

6. The function *f*, defined on the set of ordered pairs of positive integers, satisfies the following equations:
$$f(x,x) = x$$
$$f(x,y) = f(y,x)$$
$$(x+y)f(x,y) = yf(x,x+y)$$
Calculate $f(14,52)$.

7. A convex polyhedron has for its faces12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

8. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer *n* for which *n*! can be expressed as the product of $(n-3)$ consecutive positive integers.