Math Wrangle MAA - Golden Section, February 25, 2012

1. In a triangle ABC, $\angle A = 120°$. Suppose that D is a point on the angle bisector of the angle A, and $AD = AB + AC$. Find the angles CBD, BCD, and BDC.

Solution: Let K be a point on ray AD such that $AK = AB$. Triangle BAK is isosceles with $\angle BAK = 60^0$. Thus ΔBAK is equilateral and hence $\angle ABK = \angle AKB = 60°$. Therefore $\angle BKD = 180° - 60° = 120°$. $AD = AB + AC = AK + KD$, and since $AK = AB$, so $KD = AC$. We also know that $BK = AB$. Hence $\Delta KBD = \Delta ABC$, and so $\angle KBD = \angle ABC$. But then $\angle CBD = \angle KBD + \angle CBK = \angle ABC + \angle CBK = \angle ABK = 60^0$, and since $BD = BC$, ΔDBC is equilateral. Thus each angle in question is 60^0.

2. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

Solution: Let $(x + y) = u$, and $xy = w$. Then we have:

$$x^2 + y^2 = (x + y)^2 - 2xy = u^2 - 2w = 7$$
$$x^3 + y^3 = (x + y)^3 - 3xy(x + y) = u^3 - 3wu = 10$$

Solving the top equation for w and substituting it in the bottom equation we get

$$u^3 - 21u + 20 = 0$$

Obviously, $u = 1$ satisfies this equation. Dividing the cubic polynomial by $(u - 1)$ and solving the resulting quadratic equation we find all three roots: $u = 1, 4, -5$. Thus the largest value of $x + y$ is 4.

3. Does there exist a trapezoid with the property that the (positive) difference of the lengths of its sides is bigger than the (positive) difference of the lengths of its bases?

Solution: Suppose that $ABCD$ is such a trapezoid, with the longer base AB and the longer side AD. Let P be a point on AB such that CP is parallel to AD. $APCD$ is a parallelogram, and hence $PA = CD$, and $PC = AD$. Then we must have:

$$BP = AB - PA = AB - CD < AD - BC = PC - BC,$$

which contradicts the triangle inequality. Thus such a trapezoid cannot exist.

4. Austin takes red and black cards out of a bag and arranges them on a table into two stacks. It is prohibited to place a card on top of a card of the same color. The 10th and 11th cards placed by Austin on the table are both red, while the 25th card is black. What color is the 26th card placed on the table?

Solution: Note that the states with two top cards of the same color and with the two top cards of different colors alternate. Since the 10th and 11th cards are both red, the top cards after placing the 11th card are of the same color. Hence after placing the 25th card, the top cards must also be of the same color, and since the 25th card is black, they are both black. Hence the 26th card must be red.

5. What is the largest positive integer n for which there is a unique integer k such that $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$?

Solution: $8n + 8k < 15n$, and $13n < 7n + 7k$, so $\frac{6}{7}n < k < \frac{7}{8}n$. For k to be unique, we must have $\frac{7}{8}n - \frac{6}{7}n = \frac{n}{56} \leq 2$. Thus the largest possible $n = 112$.

6. The function f, defined on the set of ordered pairs of positive integers, satisfies the following equations:

$$f(x,x) = x$$
$$f(x,y) = f(y,x)$$
$$(x+y)f(x,y) = yf(x,x+y)$$

Calculate $f(14,52)$.

Solution: Let $x + y = z$. Then $y = z - x$, and the last equations yields

$$f(x,z) = \frac{z}{z-x} f(x,z-x).$$

This equation holds as long as $z - x > 0$. Similarly, if $z - 2x > 0$, then

$$f(x,z-x) = \frac{z-x}{z-2x} f(x,z-2x)$$

Hence $f(x,z) = \frac{z}{z-x} \cdot \frac{z-x}{z-2x} f(x,z-2x) = \frac{z}{z-2x} f(x,z-2x)$. Clearly,

$$f(x,z) = \frac{z}{z-nx} f(x,z-nx)$$

for any n such that $z - nx > 0$.

Thus $f(14,52) = f(14, 3 \cdot 14 + 10) = \frac{52}{10} f(14,10) = \frac{52}{10} f(10,14) = \frac{52}{10} \cdot \frac{14}{4} f(10,4)$

$$= \frac{52}{10} \cdot \frac{14}{4} \cdot f(4, 2 \cdot 4 + 2) = \frac{52}{10} \cdot \frac{14}{4} \cdot \frac{10}{2} f(2,4) = \frac{52}{10} \cdot \frac{14}{4} \cdot \frac{10}{2} \cdot \frac{4}{2} f(2,2) = 364.$$

7. A convex polyhedron has for its faces12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

Solution: The number of segments joining the vertices of the polyhedron is $\binom{48}{2} = 1128$. We must now subtract out those segments that lie along an edge or a face.

Since every vertex of the polyhedron lies on exactly one vertex of a square/hexagon/octagon, we have that $V = 12 \cdot 4 = 8 \cdot 6 = 6 \cdot 8 = 48$.

Each vertex is formed by the intersection of 3 edges. Since every edge is counted twice, once at each of its endpoints, the number of edges is $E = \frac{3}{2}V = 72$.

Each of the segments lying on a face of the polyhedron must be a diagonal of that face. Each square contributes $\frac{4(4-3)}{2} = 2$ diagonals, each hexagon $\frac{6(6-3)}{2} = 9$, and each octagon $\frac{8(8-3)}{2}$ =20. The number of diagonals is thus $2 \cdot 12 + 9 \cdot 8 + 20 \cdot 6 = 216$.
Subtracting, we get that the number of space diagonals is $1128 - 72 - 216 = 840$.

8. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $(n-3)$ consecutive positive integers.

Solution 1: The product of $n-3$ consecutive integers can be written as $\frac{(n-3+a)!}{a!}$ for some integer a. Thus, $n! = \frac{(n-3+a)!}{a!}$, from which it becomes evident that $a \geq 3$. Since $(n-3+a)! > n!$, we can rewrite this as $\frac{n!(n+1)(n+2)\ldots(n-3+a)}{a!} = n!$, and hence $(n+1)(n+2)\ldots(n-3+a) = a!$. For $a = 4$, we get $n + 1 = 4!$ so $n = 23$. For greater values of a , we need to find the product of $a - 3$ consecutive integers that equals $a!$. n can be approximated as $\sqrt[a-3]{a!}$, which decreases as a increases. Thus, $n = 23$ is the greatest possible value to satisfy the given conditions.
Solution 2: Let the largest of the (n - 3) consecutive positive integers be k. Clearly k cannot be less than or equal to n, else the product of (n - 3) consecutive positive integers will be less than $n!$.

Key observation: Now for n to be maximum the smallest number (or starting number) of the (n - 3) consecutive positive integers must be minimum, implying that k needs to be minimum. But the least $k > n$ is (n+1).
So the (n - 3) consecutive positive integers are 5, 6, 7…, n+1.
So we have $(n+1)!/4! = n!$, whence $n+1 = 24$, and so $n = 23$.

Generalization: Largest positive integer n for which n! can be expressed as the product of (n - a) consecutive positive integers is $(a + 1)! - 1$.

For example, largest n such that product of (n - 6) consecutive positive integers is equal to $n!$ is $7! - 1 = 5039$.

Proof: Reasoning the same way as above, let the largest of the (n - a) consecutive positive integers be k. Clearly k cannot be less than or equal to n, else the product of (n - a) consecutive positive integers will be less than $n!$.

Now, observe that for n to be maximum the smallest number (or starting number) of the (n - a) consecutive positive integers must be minimum, implying that k needs to be minimum. But the least $k > n$ is (n+1).

So the (n - a) consecutive positive integers are a+2, a+3, … n+1.
So we have $(n+1)! / (a+1)! = n!$, thus $n+1 = (a+1)!$, and so $n = (a+1)! - 1$.