Curvature and the Gauss-Bonnet Theorem

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Parts of this handout are taken from *Geometry and the Imagination* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston. See http://www.geom.uiuc.edu/docs/doyle/mpls/handouts/handouts.html

This unit is appropriate for high school students, preferably with knowlege of radian measure of angles, but it could be adapted to use degrees for younger students. Guidelines for students are in plain font; answers and instructor notes are in italix.

1 Introduction

A piece of paper is flat. The surface of a watermelon is curved. Our goal is to define and quantify curvature in several different ways and to discover patterns and relationships. It turns out that several different ways of computing curvature all give the same answers, but you will have to take this mostly on faith; we will not prove the equivalence of definitions here. Curvature is a major topic in the mathematical subject of Differential Geometry.

2 Swimming pool analogy

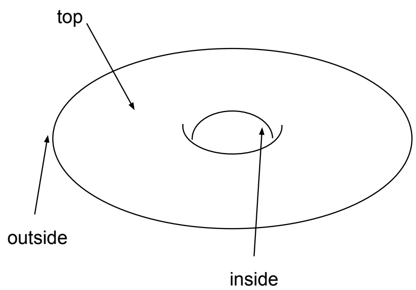
Suppose you have a swimming pool with an oddly shaped, slanted bottom and you want to talk about the amount of water in the pool. If you are interested in the amount of water below a specific point on the surface of the pool, you will be asking about the depth of the pool at that point, which is measured in units like feet or meters. If, instead, you want to know the amount of water below a small region of the surface of the pool, you will be asking about volume, measured in units like cubic feet or cubic meters. This volume can be approximated as the depth below some point in the small region times the area of the region. Finally, if you want the water in the entire pool, you again will be asking about volume. If you can't calculate it conveniently by draining the entire pool, you could estimate it by dividing the surface into many small regions, approximating the volume below each small region, and adding these volumes up. Students who are familiar with calculus may recognize that what we are really doing here is integrating: we integrate the depth of the pool over the surface of the pool.

When we talk about curvature of a surface, there are also three ways to describe it, which I will call pointwise curvature, regional curvature, and total curvature. Pointwise curvature refers to the curvature at a point of the surface, analogous to the depth of water below a single point. Regional curvature refers to the curvature within a small region of the surface, analogous to the volume of water below a region of the swimming pool. We will see that regional curvature and pointwise curvature have different units, just like depth and volume. Finally, total curvature refers to the curvature of the entire surface, which, analogous to the swimming pool, can be found by adding up regional curvatures or by integrating pointwise curvature over the entire surface.

3 Intuition

1. Which would you say is more curved, a piece of the surface of an orange or the same size piece of the surface of a watermelon?

2. Examine the surface of a bagel. How does the curvature on the inside of the hole compare to the curvature on the outside or on the top?



4 Curvature of Curves in the Plane

Before we define curvature of a surface, let's drop down a dimension and think about curvature of 1-dimensional curves.

3. Graph the parabola $y = x^2$. What part of the parabola has the biggest curvature? The smallest curvature?

   One intuitive way to think of curvature at a point on a curve is to imagine driving along the curve. The greater the curvature, the harder you would have to turn the steering wheel to stay on the road.

4. Graph the function $y = \sin(x)$ and describe its curvature at the peaks, the troughs, and at the points midway between peaks and troughs.

5. Draw a spiral. Describe the curvature at various points along the curve.

6. Draw a curve that has the same curvature at all points. How could you quantify its curvature as a number?

   A straight line has the same curvature at every point (0 curvature), but a more interesting example is a circle.

   For points on a circle, curvature can be defined as the angle you turn through per distance you travel. This leads to the formula

   $$\textit{curvature on a circle of radius } r = \frac{2\pi}{2\pi r} = \frac{1}{r}$$

   since you turn through an angle of 2π radians when you traverse the entire circumference of $2\pi r$. This definition agrees with our intuition that a small circle has greater curvature at each point than a large circle (you would have to turn the steering wheel harder to stay on the road for a small circle). This definition also agrees with our intuition that a straight line has 0 curvature, since a straight line can be thought of as the limit of larger and larger circles, and $\frac{1}{r}$ gets close to 0 as r gets very big.

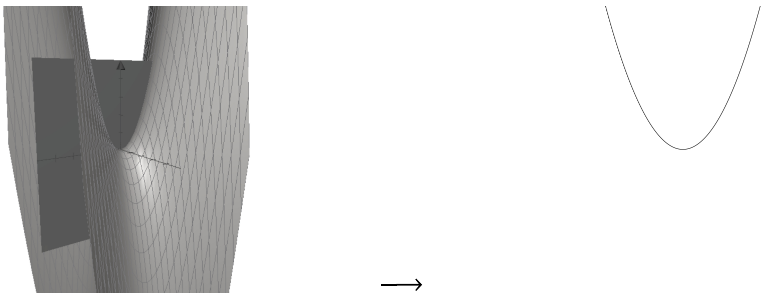
7. Extend this idea to define the curvature at points of other curves, like the parabola.

   One way to define curvature at a point of any other smooth curve is to find the radius of the circle that best approximates the curve at that point and calculate curvature as $\frac{1}{radius}$. This intuitive idea of curvature for curves in a plane is all that is needed here. A precise definition is possible using calculus. See, for example, Calculus and Analytic Geometry, 5th edition *by Stein and Barcellos, Chapter 9.*

5 Gaussian Curvature

Once we have a notion of curvature of curves in a plane, we can use this to define the curvature of a surface embedded in 3-dimensional space. Take a point of the surface, and imagine slicing the surface by a plane perpendicular to the surface at that point. The intersection of the 2-dimensional surface and the plane makes a 1-dimensional curve in the plane. This 1-dimensional curve has a curvature that we'll call a *slice* curvature of the surface.

If the surface bends in opposite directions, as happens at the saddle point below, we will call the slice curvatures in one direction positive, and the slice curvatures in the other direction negative.



At any point, there could be infinitely many slice curvatures, depending on which direction you slice the surface with a plane. At any point of the surface, the maximum and minimum slice curvatures are called the *principal curvatures* at that point. It is a fact (that we won't prove) that the principal curvatures always lie in perpendicular directions.

One way to get a single number for the curvature of a point on the surface is to multiply together the principal curvatures. This product is called the *Gaussian curvature* at that point on the surface. If the surface bends in opposite directions, as happens at a saddle point, then one principal curvature will be positive and one negative, so that the product is negative.

8. According to this definition, is the curvature at a point on a watermelon bigger or smaller than the curvature at a point on an orange?

9. Estimate the curvature at various points on a bagel. Where is the curvature greatest and least? Where is the curvature positive, negative, and zero?

   Curvature is positive on the outer part, and negative for points inside the hole, where the principal curvatures are in opposite directions. Curvature is 0 on the top and bottom of the bagel, where it is flat in one direction.

10. What is the curvature of a point on a cylinder?

11. What is the curvature of a point on the side of a cone?

    A cylinder, and a cone have zero curvature. Our current definition of curvature doesn't work for computing the curvature of the tip of the cone, since this sharp point can't be approximated by circles. However, the definition of regional curvature in a later section will allow us to handle regions containing this point.

12. What are the units of curvature?

    If the units of length used are centimeters, then the units of curvature will be $\frac{1}{cm} \cdot \frac{1}{cm}$ *or* $\frac{1}{cm^2}$*. Note that if millimeters are used, instead of centmeters, then the curvature will be 100 times smaller, since, for example, the curvature of a ball of* $25/cm^2$ *converts to* $\frac{25}{cm^2} \cdot \frac{cm}{10mm} \cdot \frac{cm}{10mm} = 0.25/mm^2$*. This agrees with our intuition that the ball would appear much less curved if we shrunk ourselves down to ant size.*

13. Calculate the curvature at each point of a sphere of radius 9 cm.

14. Estimate the regional curvature of the upper hemisphere of a sphere of radius 9 cm. What are the units of regional curvature?

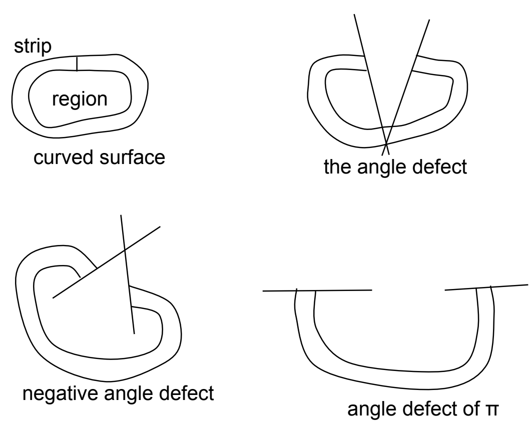
    The curvature at each point of a sphere of radius r is $\frac{1}{r^2}$*. Since the hemisphere has area* $2\pi r^2$*, the regional curvature is* 2π*. The units cancel, so regional curvature is unitless!*

6 Angle Defect and Curvature

If you take a piece of the skin of a sphere, you can't flatten it onto a plane without either stretching it or tearing it. If you've spent much time flattening orange peels, then you probably already know this. Even a small piece needs to be ripped to flatten on the table.

One way to measure the curvature of a region of a surface is to cut a narrow ring from the boundary, cut the ring open into a strip and flatten this strip onto the table, so that it opens up. The total curvature of the piece of surface is the angle by which the strip opens up, which is also called the *angle defect*. We can call this definition of curvature the *angle defect regional curvature*, where the word regional is to emphasize that it is a measure of curvature in a region of a surface, not just at a single point.

If the strip meets up with itself perfectly, then the region has zero angle defect regional curvature. Sometimes, the strip doesn't meet up because it doesn't curl enough. This is positive curvature. Sometimes the strip doesn't meet up because it curls around too much and overshoots. This is negative curvature.



It is best to measure angle defect in radians instead of degrees, so that the angle defect definition of regional curvature will agree with the previous definition based on circles. Note also that the region must have the "topology of a disk" for the angle defect definition to work. In other words, the region should not contain any holes or handles. So a small piece of the surface of a bagel is fine, but a large piece that contains the entire hole in the middle is not okay.

15. What is the angle defect regional curvature of a region of a flat piece of paper?

16. What is the curvature of a piece of the surface of a cylinder?

    Anything made a flat piece of paper (without cutting or taping together) will have regional curvature 0. Note that a strip that belts the cylinder is not allowed, since the region it contains is not disk-like, but instead, contains a hole at the top or bottom of the cylinder.

17. Measure the curvature of some of these vegetables and fruits by cutting and flattening strips that surround small regions.

    - cabbage
    - kale
    - lettuce
    - orange peel
    - banana peel
    - potato peel

You'll need to pay attention not only to the angle, but also to how the strip curls around, keeping in mind that zero curvature is a strip that comes around and meets itself. Be careful about π's and 2π's.

Round vegetables like cabbage and potatoes have positive angle defect (so positive curvature), while curly vegetables like kale have negative angle defect (so negative curvature). A banana has some regions of positive curvature but also has saddle-shaped regions of negative curvature.

One nice feature about the angle defect regional curvature is that it is possible to measure curvature even at regions of a surface that are not smooth, like the cone points on a cone or vertices on a polyhedron.

18. Make a cone and measure the curvature of a small region of the cone that contains the tip of the cone, called the "cone point". Does it depend on the shape of the cone?

    The curvature in a region containing the cone point equals 2π minus the angle of the piece of circular paper used to build the cone. Pointier cones will have larger curvature.

19. On a cube, what is the curvature of a region containing one vertex?

    $\frac{\pi}{2}$

20. Construct a surface from equilateral triangles by putting seven triangles around each vertex. What is the curvature of a piece of this surface containing one vertex?

    $-\frac{\pi}{3}$

7 Total curvature

We can calculate total curvature of an entire surface by adding up the regional curvature of a bunch of small disjoint regions that cover the surface.

For a polyhdedron with flat sides, like a cube or a dodecahedron, the curvature of any region that doesn't contain a vertex is 0, so we really only need to add up the angle defect around each vertex. An easy way to do this is to add up the angles at the corners of the faces that meet at the vertex and subtract from 2π. For instance, at any vertex of a tetrahedron there are three angles of $\frac{\pi}{3}$, so the angle defect is $2\pi - \frac{3\pi}{3} = \pi$.

21. What is the sum of the interior angles of a convex polygon (in the plane) with n sides?

    The answer is $\pi(n-2)$. Some students may calculate this by placing a point in the interior of the polygon and dividing the polygon into n triangles by connecting the interior point to each vertex with a line segment. The center angles add to 2π. Since the interior angles, together with the central angles, form all the angles of n triangles, the interior angles must add to $\pi n - 2\pi = \pi(n-2)$.

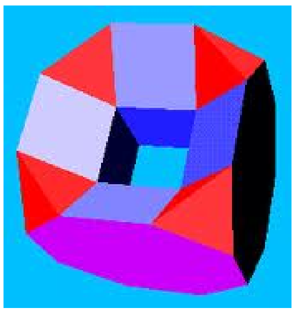
    Another way to find the answer is to imagine driving around the perimeter of the polygon in the counterclockwise direction. At each vertex, you turn to the left through an angle equal to $\pi -$ the interior angle at that vertex. Once you finish the circuit, you have gone a full 2π, so $n \cdot \pi$ - the sum of all the interior angles = 2π, and the formula follows.

22. Calculate the total curvature of the polyhedra listed below by adding up the angle defect around each vertex to get the "total angle defect".

(Semi) regular polyhedron	Angle defect of each vertex	Number of vertices	Total angle defect
Tetrahedron	π	4	4π
Triangular prism			
Cube			
Octahedron			
Cube			
Icosohedron			
Dodecahedron			
Soccer ball			

Students will notice that the total angle defect in each case is 4π.

23. Calculate its total angle defect for the polyhedron below that each form the shape of a torus (a donut).



Answer: 0

The total angle defect is intimately connected with another number from topology: the Euler number. The Euler number χ is defined as $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces. What is the relationship between total angle defect and Euler number? (This relationship is called Descartes Angle Defect Theorem.)

Descartes Angle Defect Theorem states that the total angle defect is equal to $2\pi\chi$, *where* χ *is the Euler number.*

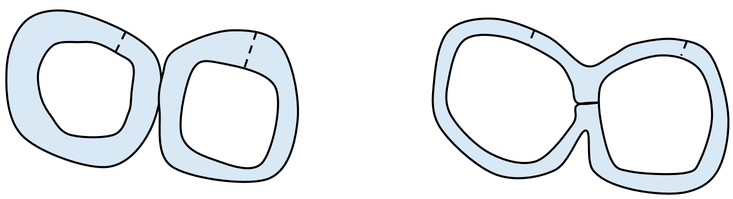
It is a fact that will not be proved here that any polyhedral division of a sphere has the same Euler number of 2, and any polyhedral division of a torus or doughnut shape has the same Euler number of 0, and in general $V - E + F$ *only depends on the "topology" of the surface, and not on how it is divided into edges, faces, and vertices.*

It is possible to prove Descartes Angle Defect Theorem using a combinatorial argument, as follows. Students who are already familiar with Euler number may appreciate this proof. This proof can also be omitted in favor of the intuitive argument using the additivity of angle defect for adjacent regions that follows.

Proof of Descartes Angle Defect Theorem: The total angle defect at all vertices is $2\pi V$ *minus the sum of the interior angles of all the polygon faces. Since the sum of the interior angles of a single polygon with* n *sides is* $\pi(n-2) = \pi n - 2\pi$, *the total angle defect at all vertices must be* $2\pi V - (\pi 2E - 2\pi F)$. *The term* $\pi 2E$ *comes from the fact that counting up all the sides of all the polygon faces counts each edge of the polyhedron twice. Simplifying, we find that the total angle defect is* $2\pi(V - E + F) = 2\pi\chi$, *as wanted.*

24. If you take two adjacent pieces of a surface, is the total curvature in both pieces put together is the same as the sum of the curvature in each piece? Why?

Notice that if you flatten out a strip around a region by making two cuts instead of one, then the sum of the two angular gaps is equal to the measure of the single angular gap you would get by flattening the strip with one cut. To calculate the curvature in two adjacent regions, surround the regions by a strip that is made up of the boundary strips for each region, put together. Flatten this strip by making two cuts, one in the boundary of each of the two regions, and add the two angle defects together.



Because angle defect is additive, it is possible to take a shortcut when calculating total curvature of polyhedra by using larger regions that contain many vertices. The same idea can be used to calculate total curvature of a round or bumpy sphere.

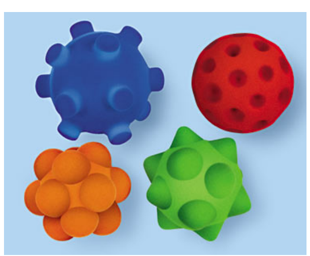
25. Use a strip around the equator to calculate the curvature in the northern hemisphere of a round sphere? The southern hemisphere?

Each half has curvature 2π. If students have trouble seeing why an equatorial strip has an angle defect of 2π, it may be helpful to start with a strip that meets up perfectly with an angle defect of 0 and gradually spreading apart the edges through an angle of $\frac{\pi}{2}$, π, and finally 2π.

26. What is the curvature on a region of the sphere that is *outside* of a tiny circle?

The same spreading process can be continued to show that it is 4π. Since the curvature inside a tiny circle is essentially 0 if the circle is tiny enough, so the total curvature of the sphere is 4π.

27. What is the total curvature of each of the surfaces shown below?



They are all 4π,no matter how bumpy or deformed the sphere is, by the same argument as above.

8 The Gauss Bonnet Theorem

The Gauss Bonnet Theorem generalizes Descartes Angle Defect Formula to surfaces that are not polyhdera. It says that the total curvature of any closed surface is $2\pi\chi$, where χ is the Euler number of the surface.

The Gauss Bonnet Theorem is amazing because it relates curvature (geometry) to Euler number (which depends only on topology). It tells us that even if earthquakes build new mountains on earth, creating additional positive curvature, that new positive curvature has to be exactly balanced by new saddle points, with negative curvature, so that the total curvature remains unchanged.

28. If a new hill creates positive curvature on a previously flat plane, where is the compensating negative curvature?

    Look for saddle shapes where the hill meets the plane.