Chomp the Graph
A Mathematical Game of Strategy

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http://youtu.be/i0s-k6ES-VM

"Even the simplest of games can pose tough mathematical challenges. One such game is Chomp". Although Ivars Peterson (2003) was referring to the game of Chomp the Chocolate, this same quote could have easily been about Chomp the Graph. Even the simple graphs which are addressed in this activity have interesting and challenging patterns which give rise to winning strategies. Starting with a list of definitions, and the rules of Chomp the Graph, this handout will explain the winning strategy for linear graphs, trees, cycles, forests, bipartite graphs, and complete graphs.

Definitions

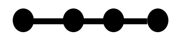
A *graph* is a finite set of vertices which are connected by edges. Only *simple graphs* are considered in this activity--- those that are undirected with no loops or multi-edges. That is, an edge must connect two distinct vertices; a pair of vertices can have at most one edge between them. Being undirected, the edges have no orientation with each edge identical to another. The *degree* of a vertex is the number of edges that connect to it. A *tree* is a connected graph with no cycles with a *leaf* defined as a vertex of degree one in a tree. For any tree, the number of edges, *e* is equal to one less than *v*, the number of vertices (*v-1=e)*. A *forest* is the union of one or more trees. A *cycle* is a graph that consists of a single cycle, or in other words, a set of vertices connected in a closed chain. For cycles, the number of vertices equal the number of edges (*v=e)*. Other types of graphs will be defined in the section which discusses the winning strategy for that graph.

Rules of Chomp the Graph

Chomp is a mathematical game of strategy in which two players take turns removing vertices and edges from graphs. Players move in turn rather than simultaneously and each player has complete information about the state of the game while making a move (Khandhawit). The winner of the game is the one who removes the last vertex, leaving the loser with nothing to remove. A player may choose to remove a vertex and all of its incident edges, or a single edge. Chomp is a terminating game since it is played on graphs with a finite number of vertices and edges. The total number of edges and vertices must decrease by at least one per turn. The minimum number of turns in a game equals the number of vertices in the graph, since all edges will be removed because they are connected to the vertices, and only one vertex can be removed in each turn. By removing first the edges, then the vertices, the maximum number of plays in a game equals the number of vertices plus the number of edges (*v+e)*. The maximum number of edges that can be removed in each turn equals the highest degree of any vertex.

Chomp is an impartial game. Both players have full knowledge of the state of the game and all moves, which depend on the state of the game, are available to both players. Chess, for example, is not an impartial game since each player can only move pieces of one color. In Chomp, the only difference for the two players is that one player goes first. For convenience, in this paper, Player A will go first, with Player B following.

A winning strategy is a sequence of moves that forces a win, no matter what moves the opponent makes (Stewart). The concept of strategy involves all possible games and is a winning one if you can make some move that places your opponent in a losing position. A losing position is one in which every move you make places your opponent in a winning position (Stewart).

Linear Graphs 

A Linear graph is a tree with every vertex having degree one or two and for our purposes, drawn in a straight line.

Figure 1 illustrates the possible play for a tree with two vertices and one edge with the image on the left showing the graph at the start of play. The second column shows the two possible types of opening moves for Player A. He may choose a vertex and the attached edge  , or may choose to remove the edge . Since the graph is undirected, we note that removing the left vertex and the adjacent edge is the same type of move as removing the right vertex and the adjacent edge. Even though they are two distinct moves, throughout this paper we will consider the possible moves up to symmetry, treating both options as the same play. The third column shows the options for Player B, and the final column is the one winning move for Player A. Although this is a short game, in all illustrations following the process of the game continues from left to right with play alternating between the two players. By choosing the edge at the beginning of the game, Player A leaves the two vertices remaining; Player B must choose one, leaving the last one for A to remove. While there are two possible ways for the game to play out, and each player wins one, since Player A goes first, he has the advantage. A winning strategy for Player A is to remove the edge on the first play, which leaves Player B in a losing position.

Figure 1

For a tree with three vertices, and 2 connecting edges, Player A has three possible types of opening moves (Figure 2).

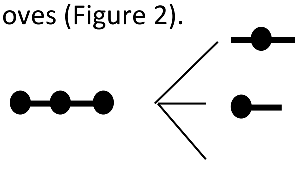
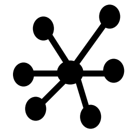


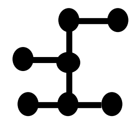
Figure 2

When Player A chooses the middle vertex, and its incident two edges this leaves two remaining vertices, and Player B has no choice but to choose one of them. If Player A chooses either leaf vertex with its one attached edge, or just an edge alone, then B has choices which would allow him to win. However, again, A has the advantage, and, choosing the middle vertex, can win every game.

In general, for any linear (tree) graph, T, Chomp can be won by Player A. Proof: Suppose the graph T has v vertices. There will always be either a vertex or edge in the middle of the path since the total number of vertices and edges is an odd number, $v+(v-1)=2v-1$. The winning strategy is for Player A to remove the vertex or edge in the middle. This gives Player B two identical trees. Whatever move Player B makes, Player A can make the identical type of move on the other component. Repeating this process we can see that Player A will make the last move, and wins the game. Thus, for any linear graph, Player A has the winning strategy.

Trees that are Stars 

Star trees are graphs with all edges branching out from a central vertex. For every size of star, Player A can always win. When the number of vertices is even, Player A chooses one edge, which leaves Player B with an even number of both vertices and edges. We will show in detail in the following sections that this is a winning position for Player A. When the number of vertices in the star is odd, A should choose the center vertex and all incident edges, leaving Player B an even number of vertices and no edges, a winning situation for Player A.

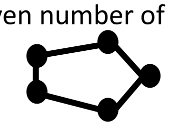
Trees of other Shapes 

In fact, Player A has a winning strategy for any tree. Although they have different shapes, and a variety of degrees for the vertices, like the Linear Graphs already discussed, Player A has the winning strategy by making a first move which leaves even vertices and edges. Unlike a Linear graph, Player A does not always mirror the play of Player B. However, he is still able to make a play which gives Player B even vertices and even edges and so continues to provide the winning strategy.

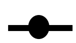
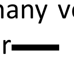
Why giving even vertices and even edges is a winning strategy

Notice that when a player removes an edge, the number of edges decreases by one and the number of vertices remains the same. When a player removes a vertex of degree *d* from any graph, then there is one less vertex, and the number of edges (*e)* decreases by *d*. A player who receives a graph with an even number of vertices cannot win the game in a single move since only one vertex can be removed in a turn.

If a player receives a graph with an even number of vertices and an even number of edges, they cannot win in just one move as mentioned above. Also, the graph that the player passes on cannot have an even number of both vertices and edges since any move either causes the edge count to go down by one or the vertex count to go down by one. A winning strategy is therefore always to pass a graph to your opponent that has an even number of vertices and an even number of edges.

Cycles 

A cycle is a connected graph with the number of vertices equal to the number of edges, *v=e*. There is a path, a sequence of edges, that visits all vertices and that does not repeat edges.

No matter how many vertices the cycle graph has, Player A has only two choices of types of opening moves:  or  , and no matter which of these he chooses, B has the winning strategy. In fact, both opening types of moves by Player A reduce the graph to a linear tree, except now the shoe is on the other foot, and B has the winning strategy.

Forests

As stated above, a forest is a graph which is the union of two or more trees. For any forest there are four cases for vertices and edges: (1) both *v* and *e* are even, (2) both v and *e* are odd, (3) *v* is even and *e* is odd, and (4) *v* is odd and *e* is even. The game of Chomp played on any non-empty forest can be won by Player B if the forest has an even number of vertices and an even number of edges and it can be won by Player A in all other cases.

Proof: Recall the above note concerning the winning strategy of even-even. If Player A receives a graph with an even number of vertices and an even number of edges, then the graph that he passes on will fall under cases 2-4 above and thus B will have the winning strategy.

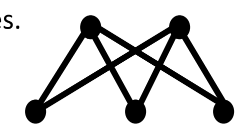
Next we consider when Player A starts with a graph in one of the cases 2, 3, or 4. As we will show below, in each of these cases Player A can always make a move that will leave their opponent with a graph that has an even number of vertices and an even number of edges. Repeating this process, their opponent will eventually be left with the empty graph and so will lose the game. In case 2, Player A is given a forest with both an odd number of vertices and edges. If the player removes a leaf, then the graph he passes to Player B will have an even number of edges and an even number of vertices, the winning strategy.

In case 3, the player is given a forest with an even number of vertices and an odd number of edges. The player can always take one edge, leaving Player B with an even number of vertices and an even number of edges.

Finally, case 4; suppose the player is given a forest with an odd number of vertices and an even number of edges. In order to give our opponent the needed even-even arrangement, we will need to remove one vertex and an even number of edges. We need to establish that there will always be a vertex of even degree.

The sum of the degrees of all the vertices in a forest equals twice the number of edges since each edge is connected on both ends to a vertex, so we know that the sum will be even. Since we know in this case that we have an odd number of vertices, we note that if each has an odd degree, adding an odd number of odd numbers will result in an odd number, and we already stated that the sum of the degrees had to be even. Therefore we have shown that there must be at least one vertex of even degree. If Player A removes this vertex, then Player B would receive a graph with an even number of vertices and an even number of edges.

We have shown that Player A can always make a move that transforms a graph in case 2, 3, or 4 into a graph in case 1. Any move Player B makes on the even-even graph will result in one of cases 2-4 again, so Player A can repeat this process in order to win (note than the empty graph is even-even). Thus, we have shown that Player B has a winning strategy if the initial forest graph has an even number of both vertices and edges, and that Player A has a winning strategy in the other cases.



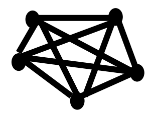
Bipartite and Complete Bipartite (Utility) Graphs

In a bipartite graph, the vertex set is the union of two disjoint sets of vertices, *W* and *X*. The top vertices in each example below belong to set *W* and the bottom vertices belong to set *X*. To be complete, as in the examples below, every vertex in *W* is connected by an edge to every vertex in *X* but there are no edges within *W* or *X*. Bipartite graphs which are not complete would be similar, except the vertices in *W* would not be connected by edges to all the vertices in *X*.

When the graph has an even number of vertices and an even number of edges, B has a winning strategy. In all other cases, A has the winning strategy. As with forests, there are four cases for vertices and edges in bipartite graphs: (1) both *v* and *e* are even, (2) both *v* and *e* are odd, (3) *v* is odd and *e* is even, and (4) *v* is even and *e* is odd. For case 1, every option for Player A leads to a winning move by Player B.

In case 2, the bipartite graph starts with an odd number of vertices and an odd number of edges. Player A would want to leave Player B an even number of vertices and an even number of edges. This means they would have to take a vertex and an odd number of edges so we will show that there will be at least one vertex with odd degree. If the bipartite graph has an odd number of vertices we can arbitrarily assign set *W* to have an odd number of vertices and *X* an even number of vertices. Since we know we have an odd number of edges, we would have to divide an odd number into an even number of vertices in *X*. This means that at least one vertex has to have an odd degree. Thus for case 2, Player A has the winning strategy by removing a vertex with an odd degree, leaving the number of vertices and the number of edges both even.

In case 3, the number of vertices is odd and the edges are even. Similar to case 2, with the graph having an odd number of vertices we can arbitrarily assign set *W* to have an odd number of vertices and *X* an even number. Since in this case the graph has an even number of edges we must have at least one vertex with an even degree. The winning strategy for this case is for Player A to choose one of the vertices with an even degree. In case 4, with an even number of vertices and an odd number of edges, all Player A has to do is select one edge for the winning strategy.



Complete Graphs

Complete graphs have the feature that each pair of vertices has an edge connecting them. In all complete graphs $v+e = (v)(v+1)/2$ (Weisstein). Notice that for a complete graph with the number of vertices equal to v, if Player A chooses to remove a vertex on his first turn, the figure turns into a complete graph with $v-1$ vertices. For example, if Player A removes a vertex and 4 edges from the pentagonal graph above, it will become a graph with 4 vertices and 6 edges.

Since a complete graph with 3 vertices is the same graph as a cycle with 3 vertices, we know Player B has the winning strategy. On a complete graph with 4 vertices, Player A should remove one vertex and the three edges connected to it. This leaves a complete graph with 3 vertices for Player B, so Player A has the winning strategy. For a complete graph with 5 vertices, Player A would not want to remove a vertex because doing so would give Player B the winning strategy.

In general, for any complete graphs, giving even-even is not always the winning strategy. Player B will win those games where $v=3n$ for some whole number, n, and Player A will have a winning strategy for all other games. Following is a summary of winning opening strategy for complete graphs:

3+3=6	*B wins*
4+6=10	*A wins by taking vertex*
5+10=15	*A wins by taking edge, 5+9=14 is winner (4+6=10 is not for B)*
6+15=21	*B wins, A takes edge 6+14=20 is not winner, but 5 +9=14 is*
7+21=28	*A wins by taking vertex*
8+28=36	*A wins by taking edge*
9+36=45	*B wins*

The general proof of this result can be found in Draisma and Van Rijnswou.

Summary of Winning Strategy for Chomp the Graph

Trees, forests and bipartite graphs have the same basic winning strategy for Chomp the Graph: give your opponent an even number of vertices and an even number of edges. Then you can match their moves to a victory continuing to leave even plus even. For trees, Player A will always have the winning strategy. For bipartite graphs and forests, some graphs will start with both even numbers of vertices and edges, which will give the winning strategy to Player B. All other graphs can be won by Player A assuming a well played beginning. Player B has the winning strategy in all games with cycle graphs since the first move of Player A turns the graph into a linear tree. Complete graphs do not follow the even plus even pattern. For complete graphs, Player A has the winning strategy in all games except those where the number of vertices is a multiple of three, then Player B has the winning strategy.

Create Your Own Math Board Game

This past spring I had my students do a project entitled "Create your own Math Board Game". To start the project we spent several class periods playing and analyzing mathematical games. We discussed what features make it mathematical, for example: using grids and graphs as the game board, measurement, estimation, fractional connections, patterns, algebraic reasoning. We also examined the logic, probability and strategies for increasing your chances of winning. A description of mathematical game features and the project rubric for "Create Your Own Math Board Game" are included with this handout. In addition, during Mathcounts club meetings, time is taken in each meeting for the team members to play games together. I encourage them to look for winning strategies, discuss whether playing first gives you an advantage, use mathematical patterns to influence their judgment of play and have fun playing the mathematical game. Chomp the Graph will be an excellent addition to the games we will play in class and meetings. It is interesting to watch the students Chomp and play other mathematical games and see if they can use their math skills to find and use a winning strategy.

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