Problem: Number Bracelets
Topic Classification: Arithmetic, Fibonacci Numbers, Dynamical Systems, Modular Arithmetic.
Level: Primary and up.
Difficulty: Easy with difficult extensions.
Prerequisites: None
Pedagogy: Exploring a question; looking for patterns.

Problem: Choose two numbers in $\{0, 1, \ldots, 9\}$ to begin a sequence of digits. The next digit in the sequence is the units digit of the sum of the two preceding it in the chain. For example if one starts with 4 and 7, the sequence starts $4, 7, 1, 8, 9, 7, \ldots$. A "number bracelet" is formed when the sequence repeats; one can think of it closing up on itself to form a loop. For example $0, 5, 5, 0, 5, \ldots$ repeats with $0, 5, 5$ as a periodic block. As a number bracelet, it has length three - the period of the sequence. The problem is to identify all possible number bracelets and their lengths.

Comments: We have used this problem in a Math Teachers Circle. The participants got involved and explored the problem at length. It was immediately engaging. One of the participants graphed successive pairs from a bracelet in the plane. The resulting pictures are worth exploring.

Modifications and Extensions: One can change the function that produces the next term in the sequence. A type of change that is relatively simple to analyze is to have $x_{n+2} = ax_{n+1} + bx_n$ for fixed a and b. One that is harder to analyze is to consider non-linear functions like $x_{n+2} = x_{n+1} + x_n^2$. In this quadratic case, the bracelet does not necessarily close up at the initial values. There can be am initial part of the sequence that is not repeated in the periodic section; this is related to the existence of square roots. One can also work modulo a different number.

References: `http://www.geom.uiuc.edu/~addingto/number_bracelets/number_bracelets.html`