Counting Cube Colorings

1. How many different ways are there to color the vertices of a cube using two colors? This question is easier to answer if the cube is not able to move. What would the answer be in that case?

2. If the cube can be moved, why is this question more difficult to answer?

3. Let's consider a simpler version of this problem. How many different ways are there to color the vertices of a square using two colors? It will be helpful to investigate the symmetries of a square, and the ways the vertices of the square can be colored if the square does not move.

4. In the top row of the chart, list all of the symmetries of a square (leave the upper left corner blank). A symmetry is a motion that rearranges the locations of the vertices and edges while returning the square to the same position it was in to begin with. The simplest symmetry is the "do nothing" motion or "identity". Be sure to create notation that makes it clear how each symmetry rearranges the vertices.

5. The symmetries of a square form a "group". A group consists of a set of objects and an operation that combines them subject to a few rules. In this case, the objects are the different motions of the square. The operation combining them is composition – make the first motion first followed by the second motion. Any two motions is equivalent to another single motion. A group must have four properties. The first is closure – whenever any two objects are combined, they must form a new object that is included in the group. The second is the existence of an identity element. When this element is combined with any other element, the result is the other element (the identity element does not change the other element). The third property is the existence of inverse elements – each element in the group must have an inverse in the group so that when the two of them are combined they result in the identity element. The last property is associativity – rearranging parentheses should not effect the outcome when three elements are combined. Commutativity (rearranging the order of two elements without changing the result) is not a required property for a group. In the case of the symmetry group for a square, some symmetries are not commutative. Can you find two elements that do not commute?

6. Along the left edge of the chart, draw pictures showing all of the different ways to color the vertices of a square using two colors if the square does not move. Try to group squares that are the same except for position next to each other in the chart.

7. Squares that are the same as each other but which are in a different position form an "orbit" under the action of the symmetry group. Use lines on the side of the chart to group the colored squares drawn on the left edge into orbits.

8. We say that a symmetry "stabilizes" a colored square in a given position if the action of that symmetry does not change the appearance of the colored square. For each row in your chart, mark the symmetries that stabilize the pictured square with an X.

9. What do you notice about the numbers of X's (stabilizers) in the various orbits? This idea is called the Orbit-Stabilizer Theorem.

10. What is the total number of stabilizers in the chart? What is the number of orbits (different ways to color a square allowing for motions)? What do you notice about these numbers? This idea is usually referred to as Burnside's lemma or the Cauchy–Frobenius lemma.

11. Try using Burnside's lemma to find the number of ways to color the vertices of a cube. Begin by counting the number of symmetries that a cube has. Remember to include the identity symmetry in your list. What is the total number of symmetries?

12. How many different ways are there to color a cube if it cannot be moved?

13. For each symmetry, think about how many different colored cubes would be stabilized by that motion. What is the total number of stabilizers in the chart?

14. Use Burnside's lemma to find the number of orbits among the colored cubes. How many different ways are there to color the vertices of a cube if it can be moved?

Orbits and Stabilizers of the Group of Symmetries of a Square
