POSSIBLE "SCRIPT" FOR A MATH CIRCLE SESSION

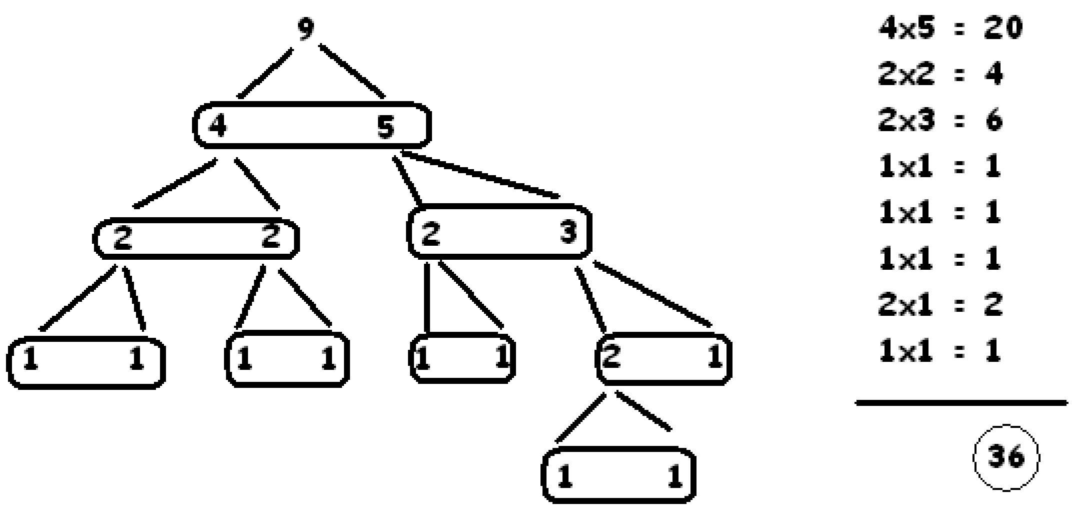
PILE SPLITTING

A 9-minute video of elements of this activity appears at
http://www.jamestanton.com/?p=825

Here's an activity which I call the "pile splitting game."

I start with a pile of 9 buttons and split it into a pile of four and a pile of five.
For no particular reason I'll write on the side of the board: $4 \times 5 = 20$.

I'll now split each of these piles in two, and write down the corresponding multiplications the side. I'll do this until I am left with nine piles each with one button.



When I add up the results of all my multiplications I get the "magic number" 36.

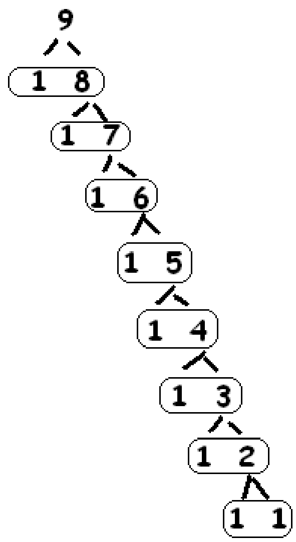
YOUR TURN: Play the game for yourself with 9 buttons, making different choices for how to split the piles. What magic number do you get?

COMMENTARY: When students try this game they each should obtain the same magic number of 36. The question of "why" now hangs in the air.

Let's play the game starting with a different number of buttons, say, 8. What magic number do we obtain now?

Keep playing with different pile sizes: 7,6,5,4,3 and 2 buttons perhaps? What do you notice about the magic numbers you obtain?

COMMENTARY: At this point someone usually notices that splitting off one button at a time explains what the magic numbers are:



1x8=8
1x7=7
1x6=6
1x5=5
1x4=4
1x3=3
1x2=2
1x1=1

36

The ninth magic number is $1+2+3+\cdots+8=36$ and the N th one shall be $1+2+3+\cdots+(N-1)$.

If you were to play the game with 101 buttons, what do you think the resulting magic number will be?

COMMENTARY: This now addresses the question, how does one compute $1+2+3+\cdots+(N-1)$? Usually some student knows the formula for this sum, but it is still worth the while developing a proof of it. Perhaps write the sum both forwards and backwards and add columnwise:

$$\begin{array}{l} 1 + 2 + 3 + \ldots + 99 + 100 = S \\ 100 + 99 + 98 + \ldots + 2 + 1 = S \\ \hline 101+101+101+ \ldots\ldots + 101+101 = 2S \end{array}$$

$$100 \times 101 = 2S$$

$$S = \frac{100 \times 101}{2}$$

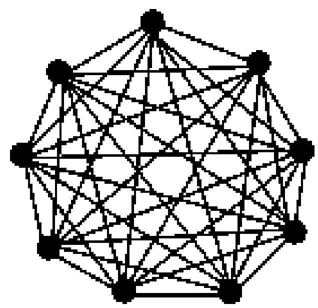
OR draw triangular arrays of dots and show that two fit together to make an $N\times(N+1)$ rectangle. Follow the lead of the students as to which approach seems best at that moment

We still haven't addressed the big question: All we've shown is that IF the magic number is always the same, then we know what that number is. But we haven't yet explained why the magic number is always sure to be the same for the same starting number of buttons.

COMMENTARY: A lot of flailing usually goes on at this point. That is okay. This is a very hard question as it stands.

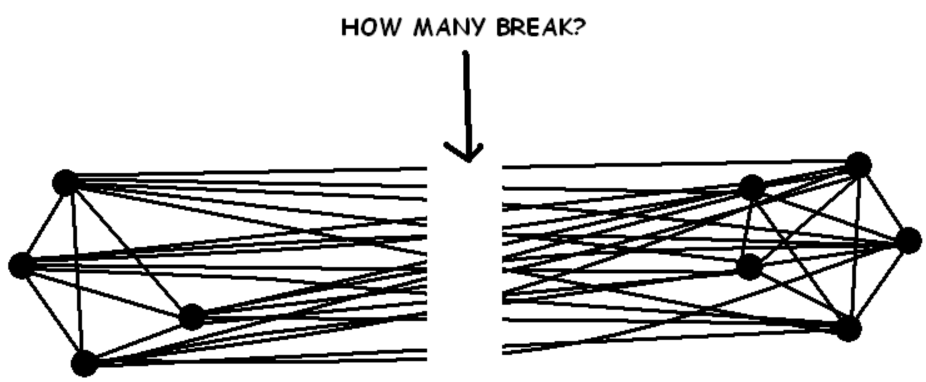
At some point I usually leap in and say something like:

Okay. Let's just change topic completely. Suppose I draw nine dots in a circle and connect each dot to each and every dot in the circle. How many lines are there?"



COMMENTARY: The students will come to see that the number of lines is either $8+7+6+...+1$ (from drawing lines from each dot in turn) or it is $(9\times 8)/2$ (from "each dots has 8 lines from it and there are 9 dots. But 9×8 double counts each line".)

Now imagine that I slide 4 of the dots to the left and 5 to the right, and the lines between dots are actually strings. When these strings are pulled across the two groups they stretch too far and break. How many strings break?



COMMENTARY: This has given a lot away! But I think it is okay for the flow of things. Usually what I have said here is enough for students to argue that the pile slitting game breaks all the strings and each multiplication on the side is just keeping track of the number of strings that break with each move. As we head towards nine piles of one button each, we break all the strings. The sum of the products must therefore equal the number of strings at the start of the game, and so be an invariant.

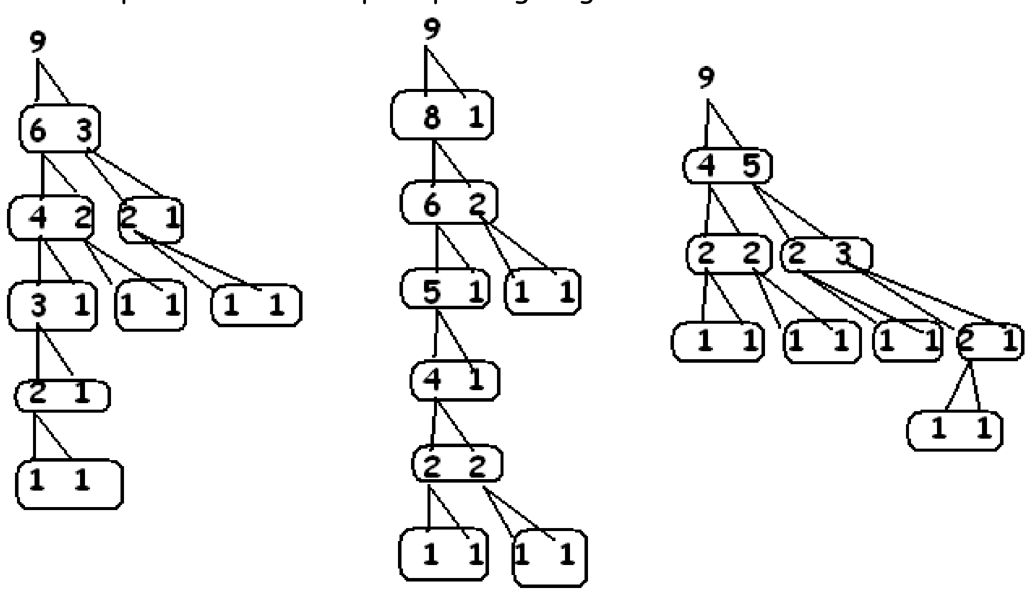
Depending on the mood of the crowd I might next ask:

(OPTIONAL): Strings connect pairs of points and break when stretched apart. Can we discover something interesting in this pile splitting game by considering sheets of triangles connecting triples of points? Imagine these triangles break when they are pulled apart.

COMMENTARY: The answer is yes. If one starts with N buttons then there are $\binom{N}{3}$ rubber triangles. If we split into a pile of a buttons and b buttons, any triangle with one dot in the left pile and two in the right, or two in the left and one in the right, breaks. There are $\binom{a}{1}\binom{b}{2}+\binom{a}{2}\binom{b}{1}=\frac{ab(a+b-2)}{2}$ such triangles. This shows that computing $ab(a+b-2)$ at each split and summing the results is an invariant! Actually, since $ab\left(a+b-2\right)=ab\left(a+b\right)-2ab$ and we know that the sum of the products ab is invariant, this means that $ab\left(a+b\right)$ is also an invariant.

Only go this route if the group seems eager for this type of technical work!!!
[And ask …**What do rubber quadrilaterals do for us?**]

Let's go back a step. Here are some pile splitting diagrams for nine buttons.



Each diagram has EIGHT pairs in it. Coincidence? Or must every pile splitting diagram for nine buttons contain eight pairs?

What about diagrams starting with N buttons?

COMMENTARY: Write nine as $1+1+1+1+1+1+1+1+1$. Each split corresponds to erasing a plus sign. There are eight plus signs and so there are sure to be eight pairs.

In general, a pile splitting diagram for N buttons is sure to contain $N-1$ pairs.

Notice that each diagram contains FOUR pairs (a,b) with each number odd. Coincidence?

What about diagrams starting with N buttons? Does the number of odd/odd pairs seem to be invariant?

COMMENTARY: I personally don't see a natural explanation of this, but maybe the students will, so be open to possibilities. Some flailing is fine.

A CHALLENGE POINT FOR THIS CIRCLE ACTIVITY:

At this point there is a split(!) in how the session might next proceed. To answer the mystery of the odd/odd pairs we need a new technique. The one I suggest opens up to a whole world of bizarre invariants and so is very exciting. It yields an exciting end to the session, but it breaks a golden rule of Math Circle work.

The second possible ending doesn't break a golden rule, but it doesn't resolve the mystery above. (It is okay to leave mysteries hanging.) This second ending has a nice approach in that shows that one can shift perspective on a topic and find a way to translate results.

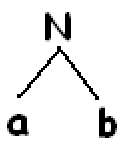
I'll describe the two endings next and in my commentary explain when I do which one with a group and why.

POSSIBLE ENDING ONE (THE EXCITING ONE)

OKAY STUDENTS … A CHANGE OF APPROACH!

Do you have a favourite formla? I do. Mine is $f(x)=\frac{1}{2}x^2$. I also like $f(x)=\frac{1}{3}x^3$.

Suppose you have a favourite formula and you decide that when you play the pile splitting game, each time you split N into $a+b$:

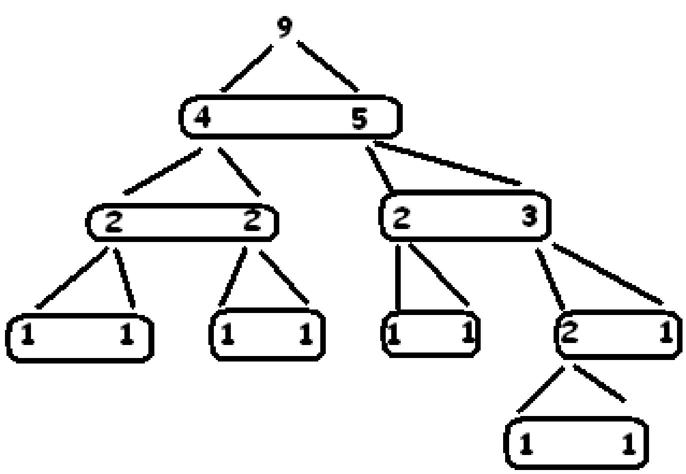


you write on the side of the page:

$$f(N) - f(a) - f(b)$$

What do you get if you add up the results?

For example:



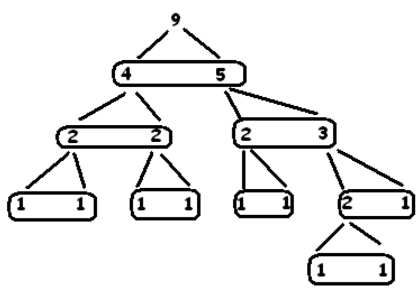
$$\begin{array}{l} f(9) - f(4) - f(5) \\ f(4) - f(2) - f(2) \\ f(5) - f(2) - f(3) \\ f(2) - f(1) - f(1) \\ f(2) - f(1) - f(1) \\ f(2) - f(1) - f(1) \\ f(3) - f(2) - f(1) \\ f(2) - f(1) - f(1) \\ \hline f(9) - 9\cdot f(1) \end{array}$$

COMMENTARY: What I have done here is delicate. I've just given a result/approach "out of thin air," which is usually deemed a bad practice in math circle work. But every now and then one needs to nudge things along in a discussion and doing this sort of prodding helps. The difference here is that with this ending I am pre-planning on making such a prod (well, a bang with something big and heavy!). Not good! This means …

> **I've conducted the circle session up to this point in such a manner that plenty of frustrating flailing has occurred - along with some successful self/group discovery. I am assuming students at this point have experienced a mix of frustration and ownership of some success.**

If I haven't succeeded in creating this mix of emotions I wouldn't be comfortable with just pulling in this "What is your favourite function?" piece right now, and probably wouldn't. I'd go with the second ending that appears after this one.

So … Having decided to go ahead with his ending, do what is implied in the table:



f(9) - f(4) - f(5)
f(4) - f(2) - f(2)
f(5) - f(2) - f(3)
f(2) - f(1) - f(1)
f(2) - f(1) - f(1)
f(2) - f(1) - f(1)
f(3) - f(2) - f(1)
f(2) - f(1) - f(1)

f(9) - 9·f(1)

Let students notice that there are lots of cancellations in the table, so many in fact that only the first quantity $f(9)$, and nine $f(1)$'s survive. The final sum is just $f(9)-9\cdot f(1)$.

Help students notice that this final sum is independent of the choices of splits made along the way!

Ask students to describe what happens in general with N buttons. [Assign to each split $N=a+b$ the quantity $f(a+b)-f(a)-f(b)$. The sum of these has many cancellations and a final, invariant, sum will be $f(N)-N\cdot f(1)$.]

Okay … Do you now see what I like $f(x)=\frac{1}{2}x^2$?

ANSWER:

$f\left(x\right)=\frac{1}{2}x^2$ means I am assigning to each split the quantity

$$\frac{1}{2}\left(a+b\right)^2-\frac{1}{2}a^2-\frac{1}{2}b^2=ab$$

The sum of the products is invariant and their final sum is always

$$f\left(N\right)-Nf\left(1\right)=\frac{1}{2}N^2-N\frac{1}{2}=\frac{N\left(N-1\right)}{2}$$

This is the opening result!

(OPTIONAL): Why do I like $f\left(x\right)=\frac{1}{3}x^3$?

ANSWER:

$f\left(x\right)=\frac{1}{3}x^3$ yields the sums $ab\left(a+b\right)$ are invariant with final sum always $\frac{1}{3}N^3-N\frac{1}{3}$.

[How does this connect with rubber triangles?]

What formula gives that the number of piles is always the same?

What formula gives that the number of odd/odd pairs are always the same?

COMMENTARY: These are tough questions and lots of good flailing is needed to get them.

$f\left(x\right)=-1$ counts the pairs (since $f\left(a+b\right)-f\left(a\right)-f\left(b\right)=1$).

It shows that the total number of pairs is always $f\left(N\right)-Nf\left(1\right)=N-1$

$f\left(x\right)=\begin{cases}-1/2 & x\ \ odd\\ 0 & x\ \ even\end{cases}$ has $f\left(a+b\right)-f\left(a\right)-f\left(b\right)=\begin{cases}1 & a\ b\ both\ odd\\ 0 & otherwise\end{cases}$ and so the number of odd/odd pairs is always $f\left(N\right)-Nf\left(1\right)=\left\lfloor\frac{N}{2}\right\rfloor$

Play! Try other formulas and see if you can create some other astounding invariants to amaze your friends!

COMMENTARY: Instead of summing terms $f\left(a+b\right)-f(a)-f(b)$ one can multiply terms of the form

$$\frac{f\left(a+b\right)}{f\left(a\right)f\left(b\right)}$$

The product is sure to be $\frac{f(N)}{\left(f(1)\right)^{n}}$, For example, choosing $f\left(x\right)=x$ shows that assigning to each split $N=a+b$ the value $\frac{f(a+b)}{f(a)f(b)}=\frac{1}{a}+\frac{1}{b}$ and taking the product of the results is sure to yield N.

Some students might think to do this sort of thing.

POSSIBLE ENDING TWO

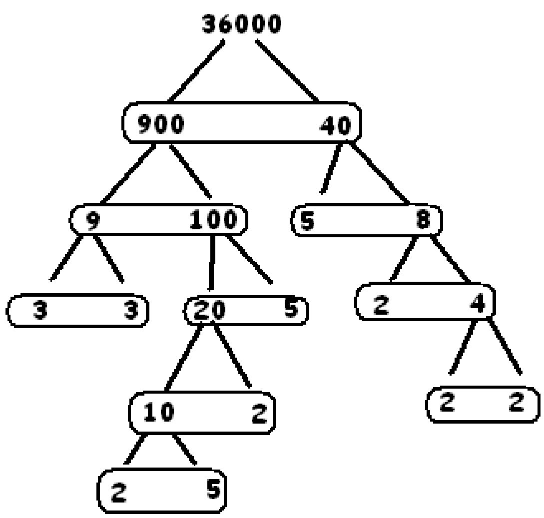
So … If it doesn't feel right to pull the function idea out of thin air, don't!

Currently we have left students with strings between dots, maybe rubber triangles between dots, and a mystery as to the count of odd/odd pairs. It is okay to leave a mystery unresolved and hanging.

Go now for a change of perspective.

Okay … Let's try a variation of pile splitting. Let's do something from grade school.

FACTOR TREES
As an example here is a tree for the number 36,000:



A factor tree diagram looks like a pile splitting diagram, and so it must have magic properties too. Can we find any?

ASIDE: When do we stop factor trees? (<u>Answer</u>: When we reach the primes.)
Why is it a good thing then NOT to consider 1 to be a prime number?

COMMENTARY: Yes. There are invariants to be found.

Have students draw different factor trees for the number 3600. Given the pile splitting they might count the number of pairs, the might write down products and sum, they might look for odd/odd pairs. Etc. Encourage them to do all that.

Here are some invariants you might want to nudge with.

1. All factor trees for 3600 contain NINE pairs of numbers.

Students can explain this by noting that 36000 is a product of ten primes and so there are nine multiplication signs between the product. Splits remove one multiplication sign at a time.

2. All factor trees for 3600 contain TWO pairs with both numbers divisible by 5.

There are three fives in the prime factorization of 36000 and they are "pulled out" one at a time.

3. All factor trees of 3600 have FOUR even/even pairs.

It is easier to explain where there are sure to be FIVE pairs with at least one odd term.

4. For each split $N \to a \times b$ write on the side of the page the product $(a-1)(b-1)$ and sum the results. For example, the tree shown gives:

$899 \times 39 = 35061$

$8 \times 99 = 792$ $\quad$ $1 \times 3 = 3$

$4 \times 7 = 28$ $\quad$ $9 \times 1 = 9$

$2 \times 2 = 4$ $\quad$ $1 \times 1 = 1$

$19 \times 4 = 76$ $\quad$ $1 \times 4 = 4$

SUM = 35,978

This sum is invariant.

This is tricky! Using the function method is best here. But students, in copying the pile splitting game might perform multiplications and try to sum them. You could nudge them to examine the products of ONE LESS than each number. This is a surprise to find this invariant.

FINAL COMMENT ABOUT FACTOR TREES
Of course there is the general question:

Is it obvious that all factor trees stop at the same set of primes?

This is big territory and only go there if you feel the mood is right and you are willing to leave the situation completely unresolved.

If you do go this route, here's our I conduct the conversation:

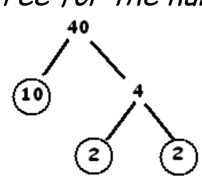
We are led to believe in grade school that all factor trees decompose a given number into the same set of primes, no matter the choices one makes along the way. This is not at all obvious. Suppose Lulu and Gary each agree to devote ten hours to factoring the number 4542847263897240000002300100000000. In what way is it transparent that they each must obtain exactly the same list of primes in the end? Here's a cute example to illustrate my point:

In the country of Evenastan only even numbers exist! If you ask a citizen of that land to count to ten, she will respond: 2, 4, 6, 8, 10. (And if you ask her to count to 11, she'll only give you a puzzled look. There is no such thing as "11" in Evenastan.)

In this world of evens, some numbers factor and some don't. For example, 24 factors (4×6) but 26 does not. (Remember, "13" does not exist.) Those numbers that factor are called e-composite (short for "Evenastan-composite") and those that don't, e-prime.

Exercise: List the first twenty e-primes.

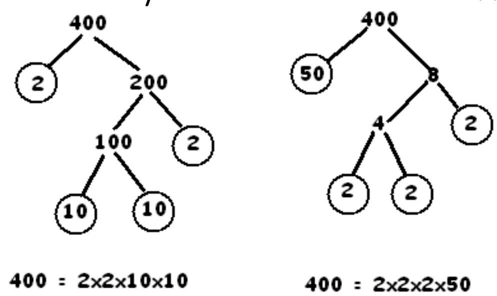
Just as for the U.S., young children in Evanastan are taught to draw factor trees. For example, here is a factor tree for the number 40:



This shows that 40 factors into e-primes: $40 = 2\times 2\times 10$.

But unlike the children in the U.S., young Evenastan scholars realize that factor trees are not unique.

For example, here are two different factor trees for the number 400 showing that *this number decomposes into e-primes in at least two different ways.*



Challenge: *What is the smallest number in Evenastan that factors into e-primes in more than one way?*

So … If factor trees are not unique in Evanastan, what makes us think they are unique in our world of evens and odds? Maybe the examples we examine in school are too small to encounter a problem?