Planar Graphs...

can be drawn in the plane as a single component with no edge crossings. Have v vertices and e edges which divide the plane into f regions called faces.
1. Euler showed $v - e + f = 2$.
2. The degree $d(v)$ of a vertex v is the number of edges which include v. Similarly the degree $d(f)$ of a face f is the number of edges in a path around its boundary.
3. Handshaking Lemma: $\sum d(v) = 2e$. For planar graphs we also have $\sum d(f) = 2e$. It follows that the number of vertices with odd degree must be even.
4. Since $d(f) \geq 3, 2e = \sum d(f) \geq 3f = 3(e - v + 2)$ so $e \leq 3v - 6$. Therefore the complete graph on 5 vertices K_5 which has $v = 5, e = 10$ is not planar.
5. If $d(f) \geq 4, 2e = \sum d(f) \geq 4f = 4(e - v + 2)$ so $e \leq 2v - 4$. Therefore the complete bipartite graph on 3 vertices $K_{3,3}$ which has $v = 6, e = 9$ is not planar (water, gas electricity problem).
6. Since $2e \leq 6v - 12 < 6v$ a planar graph must have a vertex with degree less than 6.
7. Each vertex of a planar graph can be colored with one of six colors so that no edge connects two vertices of the same color. Actually four colors suffice.
8. Five Platonic Solids. If each vertex has the same degree $d(v) = m \geq 3$ and similarly for each face $d(f) = n \geq 3$ all degrees are the same, then $mv = nf = 2e$ so $v - e + f = \frac{2e}{m} - e + \frac{2e}{n} = 2$ so $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$. The only solutions are $(m, n) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$ which correspond to the tetrahedron, cube, dodecahedron, octahedron and icosahedron respectively.
9. In any group of six people, there is a group of *three* all of which know each other or a group of *three* who don't know each other. Six is the smallest number of people for which this is true, in other words $R(3, 3) = 6$. It is difficult to prove that $R(4, 4) = 18$ and the exact value of $R(5, 5)$ is not known.