POINTS AND LINES IN THE PLANE
CENTRAL KENTUCKY HIGH SCHOOL MATHEMATICS CIRCLE

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This document contains several problems; each of these problems has a variety of answers. Your goal is to explore these questions and find as many ways to answer them as you can.

1. POINTS AND LINES...

- If there are a bunch of points in the plane, is there always a line containing exactly two of the points?

Remark 1.1. It is important for session leaders to observe that this is not a well-framed question; it is purposefully a bit vague, and requires students to massage the question to a more specific form.

The proper statement of the answer to this question is that for any collection of $n \geq 2$ points in the plane, not all lying on a line, there exists a straight line containing exactly two of the points. This is known as the Sylvester-Gallai theorem; this question was first raised by J.J. Sylvester in 1893, and the first correct proof was given 40 years later by T. Gallai. There are several nice proofs of this theorem which can be found on the wikipedia site for the Sylvester-Gallai theorem.

2. LINES AND POINTS...

- If there are a bunch of points in the plane, how many lines pass through points?

Remark 2.1. In general, this is not a simple problem, and again students will have to work to reduce the problem to a more manageable form. The best thing for a session leader to do is to help direct the students toward the following extremal variant: *Given $n \geq 3$ points in the plane, not all on a line, what is the smallest number of lines one can find that pass through at least two points?* The solution to this problem is known as the Erdős-de Bruijn theorem, which states that there are at least n such lines; this theorem was proven in 1948. The proof is by induction, and uses an application of the Sylvester-Gallai theorem, with the extremal situation realized by the "pencil" of $n-1$ points on the same line and one point off the line.

3. Colored points and lines...

- If there are a bunch of points in the plane, some colored white and some colored black, is there always a line containing exactly two of the points of the same color?

Remark 3.1. This is a "colorful" version of the Sylvester-Gallai theorem, due to G.D. Chakerian and published in 1970. Again, students will need to massage this to a more manageable form. One may get a very nice proof of this (and of the original Sylvester-Gallai theorem) through a dualizing argument that replaces points with lines and lines with points, then applies Euler's formula for planar graphs. In addition to the wikipedia article, there is a nice account of this in the chapter *Three Applications of Euler's Formula* in the book *Proofs From the Book*, by Aigler and Ziegler.

4. Colored points and convex hulls...

A set X in the plane or in three-dimensional space is called *convex* if for every pair of points in X, the line between them is also contained in X. Given a finite set P of points in the plane or in three-dimensional space, the *convex hull* of P is the smallest convex set containing P.

- What shapes can arise as the convex hull of a finite set of points in the plane? What about in three-dimensional space?
- Given three points in the plane, is it always possible to color the points white and black so that the convex hull of the white points and the convex hull of the black points intersect? What about with four points?
- Given four points in three-dimensional space, is it possible to color the points white and black so that the convex hull of the white points and the convex hull of the black points intersect? What about with five points?

Remark 4.1. This problem addresses special cases of Radon's lemma, which states that given $d+2$ points in $\mathbb{R}^d$, there is always a partition of the points into two subsets whose convex hulls intersect. Radon used this lemma implicitly in a paper from 1921. A simple proof can be found at the wikipedia site for Radon's theorem; however, the proof involves some basic linear algebra to handle the general case of convex combinations. In the two- and three-dimensional cases introduced here, a case-by-case conceptual proof should be possible.

5. Intersecting convex sets...

- Let $C_1, C_2, \ldots, C_n$ be convex sets in the plane where $n \geq 3$. Suppose that the intersection of every three of these sets contains at least one point in the plane. Do the sets $C_1, C_2, \ldots, C_n$ share a common point of intersection?
- Let $C_1, C_2, \ldots, C_n$ be convex sets in three-dimensional space where $n \geq 4$. Suppose that the intersection of every four of these sets contains at least one point in space. Do the sets $C_1, C_2, \ldots, C_n$ share a common point of intersection?

Remark 5.1. These problems are the two- and three-dimensional cases of Helly's theorem, which was first proved by Helly in 1913 and also was proved by Radon in 1921. Proofs and some history of this theorem can be found on the wikipedia site for Helly's theorem.

6. More colored points and intersecting convex hulls...

- Is it true that any set of $3(r-1)+1$ points in the plane can be colored with r different colors, where every color is used as least once, in such a way that the intersection of the convex hulls of each color class contains a point?
- Is it true that any set of $4(r-1)+1$ points in three-dimensional space can be colored with r different colors, where every color is used as least once, in such a way that the intersection of the convex hulls of each color class contains a point?

Remark 6.1. These problems are two- and three-dimensional cases of Tverberg's theorem, which was first proved by Tverberg in 1966. A current subject of research in topological combinatorics is the Topological Tverberg Conjecture, which is too complicated to state here; a partial proof of this conjecture was given in 1981 by Bárány, Shlosman, and Szücs. According to Jiří Matoušek in *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*,

> The validity of the topological Tverberg theorem for arbitrary (nonprime) p is one of the most challenging problems in this field.

If you have found this and other problems interesting, and want to know more about the current state of the art in discrete geometry, the best resource is the (warning: advanced!!) textbook by Jiří Matoušek titled *Lectures on Discrete Geometry*.