Polygons

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
October 25, 2000

It is extremely difficult to define exactly what is meant by a polygon. Everybody knows, but when you try to write down an exact mathematical definition, it is surprisingly difficult. The following definitions are pretty good, but even they are not perfect. See if you can do better.

1 Definitions

Definition 1 (Polygon) *Let $v_0, v_1, \cdots, v_n$ where $n \geq 3$ be a set of points in the plane such that $v_0 = v_n$. The union of the line segments $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$ is an n-sided polygon whose vertices are $v_1, v_2, \ldots$ and whose edges are the line segments $v_0v_1, v_1v_2, \ldots$. (Note that since $v_0 = v_n$, the polygon is automatically closed.) In what follows, let $v_i = (x_i, y_i)$ if we need to talk about the coordinates of the vertices.*

Notice that the above definition is very general—the edges of the polygon can cross each other and the vertices can coincide, or can lie on edges between other pairs of vertices.

Note that there are even more general concepts of polygons—it is possible, for example to define a polygon to be a figure that contains holes, or a polygon that has a number of disconnected components.

Definition 2 (Jordan Polygon) *A polygon P with vertices $\{v_i : 0 \leq i \leq n\}$ is said to be a Jordan polygon if:*

1. *The vertices v_i for $0 \leq i < n$ are distinct.*
2. *Two distinct edges intersect only if they have a common vertex, and they intersect only at that common vertex.*

The name "Jordan polygon" is to remind us of the famous "Jordan Curve Theorem" which states that a closed continuous curve in the plane that doesn't intersect itself divides the points of the plane into two regions called the inside and the outside. The inside has a bounded area and the outside is unbounded. Since a Jordan polygon is just a simple example of a Jordan curve that happens to be a piecewise linear curve, the Jordan curve theorem holds for Jordan polygons as well.

Definition 3 (Orientation) *A Jordan polygon is said to be oriented counter-clockwise if, as you move along the edges from v_0 to v_1 to v_2, et cetera, the inside of the curve is to your left. Otherwise, the polygon is oriented clockwise.*

Definition 4 (Convexity) *A Jordan polygon is said to be convex if the line segment connecting any two interior points is completely contained within the interior of the polygon. Polygons that do not satisfy this condition are said to be non-convex, or concave.*

Definition 5 (Star-Shaped) *A Jordan polygon is said to be star-shaped if there exists a point p in its interior such that the line segments connecting p with any other point in the interior of the polygon lie completely in the interior of the polygon.*

2 Problems

1. Show that the sum of the internal angles of a Jordan polygon with n sides is $\pi(n-2)$ (or $(n-2)180°$, if you prefer to measure angles using degrees). In other words, a triangle's internal angles add to $\pi = 180°$, those of a quadrilateral to $2\pi = 360°$, et cetera.

2. (*)Suppose you need to write computer programs to determine what kind of a polygon a set of vertices describes. What algorithms might be used to decide if the vertices form Jordan polygons, or if it's convex, or if it's oriented clockwise or counter-clockwise, et cetera? (The "(*)" indicates that the problem, or at least some portion of it, is difficult.)

3. (*)What algorithm would you use to write a computer program that will find the intersection of two convex polygons? Of two Jordan polygons?

4. Given the coordinates for a Jordan polygon P and the coordinates for a particular point p in the plane, how would you write a computer program to determine whether p is in the interior or exterior of P?

5. **Theorem 1** *(*)If a Jordan polygon has more than 3 vertices, show that it has an interior diagonal. In other words, show that there exist two vertices v_i and v_j that are not adjacent in the polygon such that the line segment v_iv_j is completely contained in the interior of the polygon.*

6. Given the result of Theorem 1, show that any Jordan polygon with n sides can be "triangulated" into $n-2$ triangles whose vertices are the same as the vertices of the original polygon.

7. Show that the vertices of the polygon in the problem above can be colored with three colors, red, green, and blue, such that each triangle in the resulting triangulation has a red, green, and blue vertex.

8. Suppose that a fancy art gallery is constructed so that its walls form a Jordan polygon when viewed from above. (We'll call this a "Jordan art gallery".) On every wall is an extremely valuable painting. Show that if the polygon is star-shaped, then a single guard can watch all the walls at once without having to move (although the guard can turn around in place).

9. Make a floor plan for an art gallery that requires more than one guard to watch all the paintings. What is the minimum number of walls needed to make a gallery that requires more than one guard? Can you design a gallery that requires 2 guards? n guards?

10. If a Jordan art gallery has n walls, what is the maximum number of guards that might be required? (Note: if our definiton for polygon had included the possibility of "holes", then this becomes an open problem—solve it and you get an instant PhD.)

11. Suppose a Jordan polygon P with n edges has vertices v_i with coordinates (x_i, y_i). Show that the (signed) area of P is given by the formula:

$$\mathcal{A}(P) = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1} - x_{i+1} y_i),$$

where the area is positive if the polygon is counter-clockwise and negative otherwise.

(Hint: Try to show it for triangles, or if that's too hard, for triangles with one vertex at the origin.)

12. **Theorem 2 (Pick's Theorem)** *If P is a Jordan polygon whose vertices all lie on lattice points (in other words, if all the coordinates of the vertices are integers), let I be the number of points completely within P, and let B be the number of points on the boundary of P. Show that the area of P is given by:*

$$\mathcal{A}(P) = I + B/2 - 1.$$

13. **Theorem 3 (Euler's Formula)** *A three-dimensional closed solid all of whose faces are polygons is called a polyhedron. Some examples include the cube, the tetrahedron, prisms, et cetera. Consider polyhedra with no holes—i.e. those that could be continuously deformed to a sphere. Let F be the number of polygonal faces, E the number of edges, and V the number of vertices. In a cube, for example, $F = 6$, $E = 12$, and $V = 8$—in other words, 6 faces, 12 edges, and 8 vertices. A tetrahedron has $F = 4$, $E = 6$, and $V = 4$. Show that for any such polyhedron:*

$$F - E + V = 2.$$