Ovals and Diamonds and Squiggles, Oh My!
(The Game of SET)

The Deck:

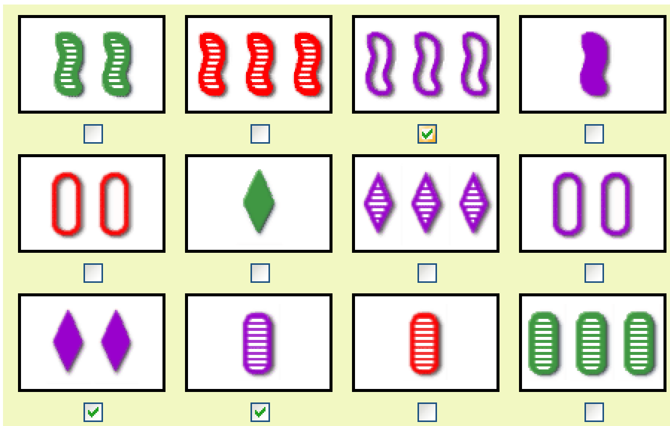
Each card in deck has a picture with four attributes

- shape (diamond, oval, squiggle)
- number (one, two or three)
- color (purple, green or red)
- shading (outlined, striped or filled in)

The full deck has one card with each possible combination of the four attributes. All the cards in the deck are different.

A Set:

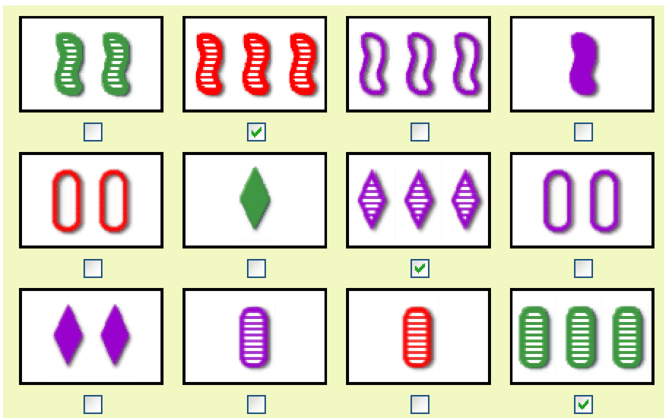
A set consists of three cards in which each attribute is either the same on all three cards or is different on each of the three cards.



In this diagram above, you see that the three checked cards form a set. Why?

1. All shapes are different
2. All numbers are different
3. All colors are the same
4. All shadings are different.

Here is another set:



Playing the game

Twelve cards are placed face up on the table. Two or more competing players look for sets. If a player finds a set, he/she says "Set" and takes the cards from the set and forms a pile. Three more cards are placed on the board and the process continues. The game continues until all the cards are dealt and no more Sets can be found. The player with the most Sets at the end of the game wins.

Special case: If the 12 cards displayed do not contain a set, three more cards are placed on the board. If a set is found, then there are 12 cards left, and players continue to look for another set, and only put down three more cards if sets can't be found.

For a simpler version, you can just use all greens.

Give it a try! Play a few rounds of the game with people sitting by you.

Questions:

1. Without counting each one, determine how many cards there are in the deck.

2. If you have two cards, how many sets can be made by adding a third card?

3. How many sets (including overlapping ones) are there? By overlapping, we mean that a given card could be a member of several sets. For example, the card with one red, striped squiggle could be a member of several sets. We want to count all possible sets.

4. What is the probability that if you pull out three cards at random from the whole deck, they form a set? To answer this question, it will be helpful to first determine how many ways there are to randomly pick three cards from a deck.

5. What are strategies for looking for sets?

6. When you were playing the game, you may have noticed that some types of sets are more likely to occur than others. Make a conjecture on which sets are more likely to occur by circling the option(s) you feel best represents the kinds of sets you were seeing most when you played:
   a. all attributes different
   b. 3 attributes different
   c. 2 attributes different
   d. 1 attribute different
   e. 0 attribute different

A conjecture is an educated guess, and we would like to do better. We will now compute the probabilities of each of these events a)-e) of occurring, so that we know exactly which is more likely to occur. The following questions will guide you to compute these probabilities.

7. We will first determine the probability that in a randomly chosen set, all attributes are different.
   a. How many ways can a set be chosen so that all four attributes are different?
   b. Given a randomly chosen set, what is the probability that all attributes are different?

8. Now we will determine the probability that in a randomly chosen set exactly three of the attributes are different. This is a bit more complicated than #7, and so what we will do is consider a simpler problem and work our way up to answering this question.

   a. Simpler scenario: Consider a reduced deck of SET cards in which one attribute is held constant. For example, suppose your deck is made up of only the green cards.
      i. How many cards are in the green deck?
      ii. How many ways can you select a set from the green deck in which all three remaining attributes are different? (Note that there are parallels between this and #7a.)

   b. Now let's go back to the full deck of SET.
      i. Determine the number of ways to select a set in which any color (whether red, green or purple) is the only common attribute.
      ii. Finally, recall that in the simpler case, we began by reducing the deck in order to have a common color. We could also have selected a deck consisting only of cards with a common shape, or common shading, etc. With this in mind, how many ways can a set be selected in which any one attribute is held constant while the remaining three attributes are different? What is the probability of randomly selecting such a set?

9. Follow some of the ideas used in #8 to help answer the following questions:
   a. Given a set, what is the probability that the cards in the randomly chosen set have *two* attributes that are different? Hint: start with a reduced deck consisting only of, say, green, striped shapes. Then consider the remaining shading styles and colors, and finally, the number of ways any two attributes (such as shape and number, or color and number, etc.) can be paired in any order.
   b. Given a set, what is the probability that the cards in the randomly chosen set have only *one* attribute that is different?
   c. Given a set, what is the probability that in a randomly chosen set, *none* of the attributes are different?

10. Now return to #6, and state which type of set is more likely to occur.

11. Competition: find the largest set-less collection of cards you can in 10 minutes.

To play on line, visit: http://www.setgame.com/set/puzzle_frame.htm

Answers:

1. Without counting each one, determine how many cards there are in the deck.
   Answer: 81. To get this, we see that we need to see how many cards can be made by changing the attributes. So the total number of cards is
   (# possible shapes)(#possible colors)(#possible shadings)(#possible numbers)= $3 \cdot 3 \cdot 3 \cdot 3 = 81$.
2. If you have two cards, how many sets can be made by adding a third card?
   Answer: Only one. Once the first two have been chosen, the color, shading, number and shape or the third is uniquely determined.
3. How many sets (including overlapping ones) are there?
   Answer: $\frac{81 \cdot 80 \cdot 1}{6} = 1080$. Here, 81 is the number of choices you have for the first card, 80 is number of choices you have for second, and 1 is number of choices you have for the third. So $81 \cdot 80 \cdot 1$ is the number of ways you could line up (or order) three cards from the deck. But a set ignores order, so this number is too large by a factor of how many ways you can arrange 3 cards among themselves. There are $3 \cdot 2 \cdot 1 = 6$ ways to arrange 3 cards among them selves, so we should divide by 6.
4. What is the probability that if you pull out three cards at random from the whole deck, they form a set?
   Answer: 1080/C(81,3)=$\frac{1080}{\frac{81 \cdot 80 \cdot 79}{3 \cdot 2 \cdot 1}} = \frac{1}{79}$, or about 1.3%. Here, 1080 is the number of possible sets and C(81,3) is the total number of combinations of 3 cards from a group of 81. Using the lingo of probability, C(81,3) is the size of the sample space and 1080 is the size of the event that your randomly chosen three cards forms a set.
5. What are strategies for looking for sets?
   Answer: will vary
6. Which sets are more likely?
   a. all attributes different - 20% of all sets
   b. 3 attributes different - 40% of all sets
   c. 2 attributes different - 30% of all sets
   d. 1 attribute different - 10% of all sets
   e. 0 attribute different - 0%

Type of set (# attributes same and different)	Ways to pick first card	Ways to pick the attribute that is the same on the 2nd card	Ways to pick the second card with given attributes chosen from ←previous column	Ways to pick the third card	Number of sets of this type	Likelihood of this type of set
4 different 0 same	81	C(4,0)=1	$2^4 = 16$	1	$\frac{81 \cdot 16}{6} = 216$	$\frac{216}{1080} = 20\%$
3 different 1 same	81	C(4,1)=4	$2^3 = 8$	1	$\frac{81 \cdot 4 \cdot 8}{6} = 432$	$\frac{432}{1080} = 40\%$
2 different 2 same	81	C(4,2)=6	$2^2 = 4$	1	$\frac{81 \cdot 6 \cdot 4}{6} = 324$	$\frac{324}{1080} = 30\%$
1 different 3 same	81	C(4,3)=4	$2^1 = 2$	1	$\frac{81 \cdot 4 \cdot 2}{6} = 108$	$\frac{108}{1080} = 10\%$

7. We will first determine the probability that in a randomly chosen set, all attributes are different.
   a. How many ways can a set be chosen so that all four attributes are different?
   $\frac{\#ways\ to\ choose\ first\ card \cdot \#\ ways\ to\ choose\ second \cdot \#\ ways\ to\ choose\ third}{3 \cdot 2 \cdot 1}$
   $= (81 \cdot 2^4 \cdot 1)$ /6=216
   Note: the number of ways to choose the second card is the same as the number of ways to pick the attributes of the second card. So the number of ways to do this is
   $$\#\ ways\ to\ pick\ color \cdot \#\ ways\ to\ pick\ shape \cdot \#\ ways\ to\ pick\ shading \cdot \#\ ways\ to\ pick\ number = 2 \cdot 2 \cdot 2 \cdot 2$$
   b. Given a set, what is the probability that all attributes are different?
   $$\frac{216}{1080} = 20\%$$

8. Now we will determine the probability that in a randomly chosen set exactly three of the attributes are different. This is a bit more complicated than #7, and so what we will do is consider a simpler problem and work our way up to answering this question.

   a. Simpler scenario: Consider a reduced deck of SET cards in which one attribute is held constant. For example, suppose your deck is made up of only the green cards.
      i. How many cards are in the green deck? 27
      ii. How many ways can you select a set from the green deck in which all three remaining attributes are different? (Note that there are parallels between this and #7a.) $(27 \cdot 2^3 \cdot 1)/6$

   b. Now let's go back to the full deck of SET.
      iii. Determine the number of ways to select a set in which any color (whether red, green or purple) is the only common attribute. $(81 \cdot 2^3 \cdot 1)/6$
      iv. Finally, recall that in the simpler case, we began by reducing the deck in order to have a common color. We could also have selected a deck consisting only of cards with a common shape, or common shading, etc. With this in mind, how many ways can a set be selected (from a complete deck) in which any one attribute is held constant while the remaining three attributes are different? What is the probability of randomly selecting such a set? $number\ of\ ways = 4 \cdot (81 \cdot 2^3 \cdot 1)/$ 6=432, $probability = \frac{432}{1080} = 40\%$

9. Follow some of the ideas used in #8 to help answer the following questions:
   c. Given a set, what is the probability that the cards in the randomly chosen set have *two* attributes that are different? Hint: start with a reduced deck consisting only of, say, green, striped shapes. Then consider the remaining shading styles and colors, and finally, the number of ways any two attributes (such as shape and number, or color and number, etc.) can be paired in any order.
   $$\frac{81 \cdot 6 \cdot 4}{6} = 324$$
   d. Given a set, what is the probability that the cards in the randomly chosen set have only *one* attribute that is different?
   $$\frac{81 \cdot 4 \cdot 2}{6} = 108$$
   e. Given a set, what is the probability that *none* of the attributes are different? 0, since if none are different then all cards are the same, which wouldn't be a set.

10. Now return to #6, and state which type of set is more likely to occur. 3 different, 1 same

11. Competition: find the largest set-less collection of cards you can in 10 minutes.
    20 is the largest set-less pile (proved by computer using brute force)