



# THE STERN-BROCOT TREE



This is a sample (draft) chapter from:

*MATHEMATICAL OUTPOURINGS*

Newsletters and Musings from the  
St. Mark's Institute of Mathematics

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This material was – and can still be – used as the basis of a successful  
MATH CIRCLE activity.



The St. Mark's

**Institute of Mathematics**

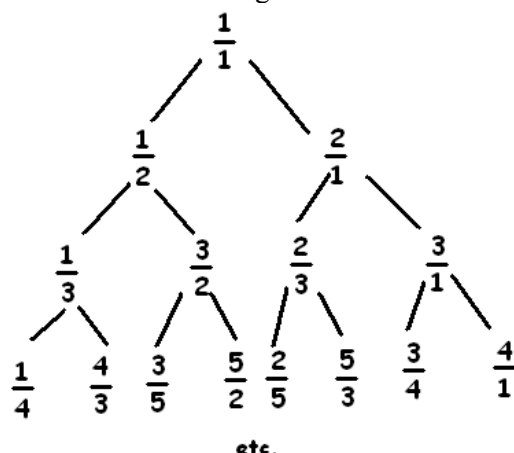
Newsletter



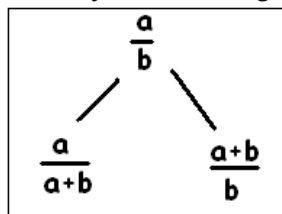
JANUARY 2010

**PUZZLER:****A CURIOUS FRACTION TREE**

Here is something fun to think about.  
Consider the following "fraction tree:"



It is constructed by the following rule:



Each fraction has two "children:" a left child, a fraction always smaller than 1, and right child always larger than 1.

a) Continue drawing the fraction tree for another two rows.

b) Explain why the fraction  $\frac{13}{20}$  will eventually appear in the tree. (It might be easier to first figure what should be the parent of  $\frac{13}{20}$  and its grandparent, and so on.)

c) Might the fraction  $\frac{13}{20}$  appear twice in the tree?

d) Will the fraction  $\frac{457}{777}$  eventually appear in the tree? Might it appear twice?

e) Are all the fractions that appear in the tree in reduced form? (For example, I see the fraction  $2/3$ , but I don't see its equivalents  $4/6$  or  $12/18$ .)

f) Let me give things away: Prove that this tree produces all possible positive fractions, written in reduced form, with no fraction ever appearing more than once.

**THE EUCLIDEAN ALGORITHM:**

If two numbers  $a$  and  $b$  are multiples of three, then so too will be their difference:  $b - a$ . In fact, it is not too hard to see that if one lists all the common factors of  $a$  and  $b$  and then lists all the common factors of  $a$  and  $b - a$ , the two lists will be the same. [For example, with  $a = 84$  and  $b = 120$  the pair  $(84, 120)$  has common factors: 1, 2, 3, 4, 6, and 12, and the pair  $(84, 36)$  has common factors: 1, 2, 3, 4, 6, and 12!]

Greek geometer Euclid (ca. 300 B.C.E) noticed this too and went further to observe that if we play the game of repeatedly subtracting the smaller number from the larger from a pair of numbers to produce new pairs, the lists of common factors cannot change:

$$(84, 120) \rightarrow (84, 36) \rightarrow (48, 36) \\ \rightarrow (12, 36) \rightarrow (12, 24) \rightarrow (12, 12)$$

*All pairs have common factors:  
1, 2, 3, 4, 6 and 12.*

ON REFLECTION: Newsletter January 2010  
twice, as would its grandparent, great grand  
parent, and so on, all the way down to  $1/1$   
appearing twice, which does not!

**COMMENT:** If we read the fractions left to  
right across the rows of the tree we obtain  
the sequence:  $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{3}, \dots$ . This  
list contains all the positive fractions. The  
fact that one can list the rationals was first  
discovered by Georg Cantor in the 1800s.  
He accomplished this feat a different way.

**CHALLENGE:** Put all the rationals, both  
the positive and negative fractions, including  
zero, into a single list.

**SOME MYSTERIES:** Look at the  
numerators that appear in the fraction tree,  
reading across the tree by its rows. We  
obtain the sequence:

**1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 5 ...**

(The denominators follow the same  
sequence offset by one.)

a) Pluck out every second entry, the ones in  
even positions. This gives 1, 1, 2, 1, 3, 2, 3,  
1, ..., the original sequence!

b) Each entry in an odd position (after the  
first) is the sum of its two neighbours!  
 $2 = 1 + 1, 3 = 1 + 2, 3 = 2 + 1, 4 = 1 + 3, 5 = 3 + 2, \dots$

c) Every third entry is even. All other entries  
are odd.

d) The entries between the ones are  
palindromes: 2 323 4352534  
547385727583745 ... . The digits of each  
of these palindromes add to one less than a  
power of three.

**RESEARCH CORNER:** Prove these  
claims. Discover more remarkable  
properties of this sequence. Can you push  
the sequence backwards? For instance, b)  
says that the entry to the left of the first 1  
must be zero. What's next on the left?

As the numbers in the pairs are decreasing in  
size (we do not allow ourselves to go to  
zero), this process must stop and do so when  
there is no smaller value to subtract from a  
larger. That is, this process must stop by  
reaching a pair with a common value:

$(d, d)$ . But the pair of numbers  $d$  and  $d$   
has the same list of common factors as do  
the original pair  $a$  and  $b$ . Since  $d$  is  
clearly the largest common factor of  $d$  and  
 $d$ , we must have that this final number  $d$  is  
the largest common factor of  $a$  and  $b$ .

#### *EUCLID'S ALGORITHM:*

*To find the greatest common factor of two  
numbers, repeatedly subtract the smaller  
from the larger until a common value  
appears. This common value is the greatest  
common factor!*

For example, consider 102 and 170. The  
algorithm gives:  
 $(102, 170) \rightarrow (102, 68) \rightarrow (34, 68) \rightarrow (34, 34)$   
Their greatest common factor is 34.

**EXERCISE:** Find the greatest common  
factor of 457 and 777.

#### **CONNECTIONS TO THE FRACTION**

**TREE:** Start at apex  $\frac{1}{1}$  of the fraction tree  
and follow a path down as far as you please  
and stop at some fraction  $\frac{a}{b}$ . Do you see

that this path is just the Euclidean algorithm  
conducted in reverse? In fact, starting at  
 $a/b$  and following your path back to  $1/1$   
is precisely the path of Euclid's algorithm  
applied to the pair  $(a, b)$ . As this reverse  
path yields the final pair  $(1, 1)$ , the fraction  
 $a/b$  has greatest common factor 1 between  
its numerator and denominator. That is,  
 $a/b$  must be in reduced form. Moreover,  
every reduced fraction must appear in the  
tree: just apply the Euclidean algorithm to  
the numerator and denominator to see where  
it lies in the tree.

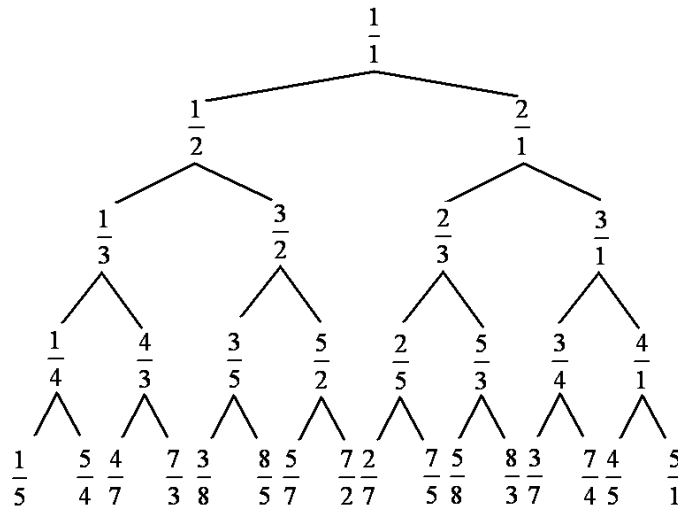
Finally, no fraction appears twice in the tree.  
If one did, then its parent too would appear

**STERN-BROCOT TREE**  
**January 2010**

**COMMENTARY, SOLUTIONS and THOUGHTS**

This remarkable fraction tree is known as the Stern-Brocot tree and it appears in [GRAHAM, KNUTH and PATASHNIK]. A similar construct, known as the Calkin-Wilf tree, is given by a very similar algorithm: each reduced fraction  $\frac{a}{b}$  has left child  $\frac{a}{a+b}$  but right child  $\frac{b}{a+b}$ . Its properties, akin to what we have outlined in the newsletter, are presented in [CALKIN and WILF]. The content of this newsletter also appears in [TANTON].

In the Fall of 2009, young students of the St. Mark's Institute of Mathematics research class also explored the properties of the Stern-Brocot tree:



They asked the following question:

*If we enumerate the fractions reading across the rows of the tree we obtain the following sequence of all positive rationals:*

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \dots$$

*What is the 100<sup>th</sup> fraction in this list? The 400,003<sup>rd</sup> fraction? Is it possible to determine the N -th fraction in this list without having to enumerate entire rows of this tree?*

Here is their result:

**Theorem 1:** Write  $N$  in base two and take note of the count of digits that appear in each block of 1s and 0s in its representation. Suppose

$$N = \overbrace{1 \cdots 1}^{a_k} \overbrace{0 \cdots 0}^{a_{k-1}} \overbrace{1 \cdots 1}^{a_{k-2}} \cdots \overbrace{0 \cdots 0}^{a_1} \overbrace{1 \cdots 1}^{a_0}$$

(with  $a_0$  set equal to zero if  $N$  is even). Then the  $N$ -th fraction  $f_N$  in the Stern-Brocot sequence is the continued fraction:

$$f_N = [a_0, a_1, a_2, \dots, a_k] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_k}}}$$

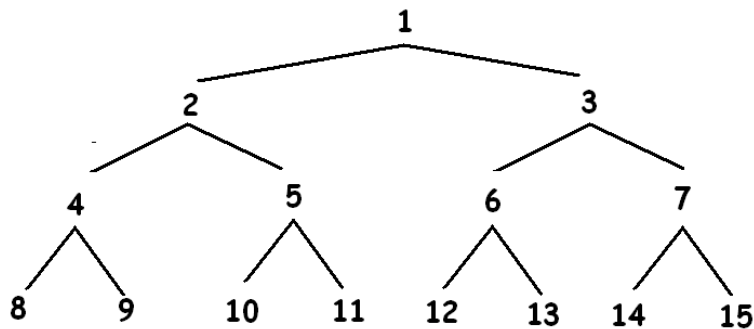
For example, the number one-hundred in base two is 1100100 and so

$$f_{100} = 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}}} = \frac{7}{19}$$

and 400003 in base two is 1100001101010000011 and so

$$f_{400003} = [2, 5, 1, 1, 1, 1, 2, 4, 2] = \frac{1553}{713}$$

**Proof:** Construct a tree of similar shape using the counting numbers:



We see that it follows the rule:



For numbers written in binary, this has a nice interpretation:

*To create the “left child” of a counting number, append a zero at the end of its binary representation. To create a “right child,” append a one.*

Thus each digit 0 in a binary representation of a number represents a left down-step in the tree and each 1 (after the first) a right down-step in the tree. The number fourteen, for example, with binary code 1110 can be interpreted as “1RRL” which means:

*Start at 1. Step right. Step right. Step left.*

In terms of blocks, if  $N = \overbrace{1\cdots 1}^{a_k} \overbrace{0\cdots 0}^{a_{k-1}} \overbrace{1\cdots 1}^{a_{k-2}} \cdots \overbrace{0\cdots 0}^{a_1} \overbrace{1\cdots 1}^{a_0}$  has blocks  $a_k, a_{k-1}, \dots, a_1, a_0$  (with  $a_0$  possibly zero), then its left child, “addend a zero,” has blocks  $a_k, a_{k-1}, \dots, a_1, a_0, 1, 0$  if  $a_0 \neq 0$  and blocks  $a_k, a_{k-1}, \dots, a_1 + 1, 0$  if  $a_0 = 0$ , and its right child, “addend a 1,” has blocks  $a_k, a_{k-1}, \dots, a_1, a_0 + 1$ .

The fractions in the Stern-Brocot tree follow the same pattern! If

$$\frac{a}{b} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_k}}}$$

then its left child is:

$$\frac{a}{a+b} = \frac{1}{1 + \frac{1}{a/b}} = 0 + \frac{1}{1 + \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_k}}}}}$$

(which equals  $\frac{a}{a+b} = \frac{1}{1 + \frac{1}{a/b}} = 0 + \frac{1}{a_1 + 1 + \frac{1}{a_2 + \cdots \frac{1}{a_k}}}$  if  $a_0 = 0$ ), and its right child is

$$\frac{a+b}{b} = \frac{a}{b} + 1 = (a_0 + 1) + \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_k}}}$$

This establishes the proof. □

Students went further and found the following additional curiosity:

**Theorem 2:** For  $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{2}{5}, \frac{5}{3}, \frac{4}{1}, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \dots$ , the sum of any fraction in the sequence and the reciprocal of its next term is always an odd integer. Precisely,  $f_{N-1} + \frac{1}{f_N} = 2a_0 + 1$  where  $a_0$  are the number of zeros at the tail end of the binary representation of  $N$ .

This provides a swift means for computing the  $N$ -th fraction  $f_N$  if  $f_{N-1}$  is already known.

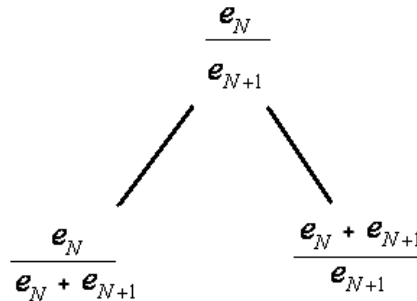
### THE NUMERATORS IN THE FRACTION TREE

Again let  $f_N$  denote the  $N$ -th fraction in the Stern-Brocot tree in reading left to right across its rows. Given the construct of the tree, it is not difficult to see that the denominator of  $f_N$  always matches the numerator of  $f_{N+1}$ . Thus if  $e_N$  denotes the numerator of  $f_N$  we have:

$$f_N = \frac{e_N}{e_{N+1}}$$

with the sequence of numerators beginning 1, 1, 2, 1, 3, 2, 3, 1, ....

Also, given the construct of the tree:



we see

$$\begin{aligned}
 e_{2N} &= e_N \\
 e_{2N+1} &= e_N + e_{N+1}
 \end{aligned}$$

This shows that the even terms the sequence  $\{e_N\}$  match the original sequence and that the odd terms satisfy  $e_{2N+1} = e_N + e_{N+1} = e_{2N} + e_{2N+2}$  as claimed in the newsletter. The palindromic properties of the sequence follow as a consequence of these relations. (The sequence starts with the palindromes 1 and 121 and these represent the next few even terms of the sequence perpetuating the symmetry. The odd terms, being the sums of neighboring terms, preserve this symmetry too.)

**ANOTHER BINARY CONNECTION**

Every positive integer has a unique binary representation using the digits 0 and 1. But if we permit use of the digit 2 as well, a number may be expressed in base two in more than one way. For example, the number ten now has five possible representations:

$$\begin{aligned} 1010 &\leftrightarrow 8 + 2 \\ 1002 &\leftrightarrow 8 + 1 + 1 \\ 210 &\leftrightarrow 4 + 4 + 2 \\ 202 &\leftrightarrow 4 + 4 + 1 + 1 \\ 122 &\leftrightarrow 4 + 2 + 2 + 1 + 1 \end{aligned}$$

I present this to young students as a weighing puzzle:

*I have two rocks each weighing 1 pound, two rocks each weighing 2 pounds, two each weighing 4 pounds, two each weighing 8 pounds, and so on. In how many ways can I make ten pounds?*

Let  $b(N)$  denote the count of ways of making a weight of  $N$  pounds. This gives the sequence  $\{b(N)\}$  known as “Stern’s diatomic sequence” [SLOANE, A002487].

If  $N$  is odd, then we must use precisely one 1 pound rock and we’re left trying to make a weight of  $N - 1$  pounds using rocks of weights 2, 4, 8, 16 .... Dividing through by two we see that this is equivalent to computing  $b\left(\frac{N-1}{2}\right)$ . Thus we have:

$$b(2k+1) = b(k)$$

If  $N$  is even, we can either use two 1 pound rocks and match a weight of  $N - 2$  pounds with the remaining rocks (and there are  $b\left(\frac{N-2}{2}\right)$  ways to do this) or we can use no 1-pound weights and match  $N$  pounds with the remaining rocks (and there are  $b\left(\frac{N}{2}\right)$  ways to accomplish this).

This gives:

$$b(2k) = b(k) + b(k+1)$$

These relations allow us to write down the sequence  $\{b(N)\}$ :

$$1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, \dots$$



and we see the sequence  $\{e_N\}$  offset by one! (Note that the recursive relations for  $e_{N-1}$  and  $b(N)$  are the same.)

Given our work with the Stern-Brocot tree, the following result is clear. But if it were presented cold, with no context for its derivation, it is truly mind boggling!

**Theorem 3:** *Consider the sequence 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ... given by counting the number of ways to represent each integer in base two using the digits 0, 1 and 2. Then every pair of coprime integers  $(a, b)$  appears in this sequence as a pair of neighboring terms exactly once.*

We can say more about this sequence by rephrasing theorem 2:

*For any three consecutive terms in this sequence, the sum of the first and last term is an odd multiple of the middle term.*

## REFERENCES:

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