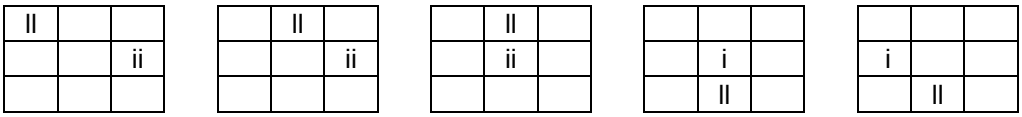
Ski Jumps

Ski Jumps is a partizan game. It is played by two players (**L**eft and R**i**ght) on a rectangular grid decomposed into squares. At the start of the game each player has a number of skiers to control.

The players take turns. Left may move one skier one square to the right (provided it is empty) or jump over one of Right's skiers (provided the left skier has two skis and is directly above the right skier, and provided the square beneath the right skier is free). If a skier with two skis is jumped, it will lose a ski.

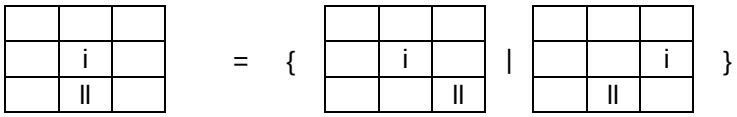
By convention, Left gets to move first. The last player to move wins. The following is a sample game:



Left now has one more move. Notice that without the jump, left would lose.

An abstract game can be represented by listing all of the possible moves for Left and all of the possible moves for Right. As an example, { | } represents a game in which Left has no moves and Right has no moves. Thus, if it is Left's turns, Left will lose, and vice-versa. With { A, B, C | D, E, F, G }, Left can move to positions A, B, or C. Each of these is also a game, and can be represented by a similar list. To understand this a bit better, consider a new game. In the nLmR game, there is a stack of L's and R's and on each turn Left or Right can remove one letter of their own type. Thus LR = { R | L } because Left can remove one L leading to a list with just R. Similarly, R = { | { | } } because Left has no move in this game and R can move by taking the R off of the list moving to the position in which neither player can move.

We have:



This looks a lot like the LR game. In fact it is equivalent to the LR game. We say a game is the zero game if whichever player plays first will lose under optimal play. We denote it by 0 or { | }. We see that LR is also a zero game. The sum G + H of G = { P | Q } and H = { R | S } is G + H = { $P \cup R$ | $Q \cup S$ }. In other words to play the sum of games G and H players take turns making legal moves in either game G or in game H. The negation of a game is obtained by having the two players switch roles. Thus, - G = { Q | P }. For example –L = R = 0L1R.

We say two games, G and H, are equivalent when G- H = G + -H is the zero game.

1. Explain why 5L2R is equivalent to 3L0R.

2. Find a game of the form nLmR equivalent to

ll		
		ii

+

ll		
		ii

3. What value of n should we choose to represent the game

ll		
		ii

?

As a project try to determine the value of every Ski Jump game on a 3 by 5 grid.

A game is impartial if the moves for Left are the same as the moves for Right, i.e. G = { *P* | *P* }. Our Die Game, and our version of Kalah are both impartial games.

The system of "numbers" described by all games in this way is very rich. Try to figure out how two games should be multiplied, what a square root would be, or a game worth exactly π. This system of numbers includes infinitesimals and ordinals. For more information see:

E. Berlekamp, J. Conway, R.. Guy, *Winning ways for your mathematical plays*, Academic Press, London, 1982, vol. 1, Games in general, vol. 2, Games in particular.

J. Conway, *On numbers and games*, Academic Press, London and New York.