Constructing voting paradoxes with logic and symmetry
Teacher's Notes

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Mobile Math Circle

This is a loose transcript of the Math Circle, with occasional notes on pedagogy. The material is roughly for three one to one-and-half hour Math Circle sessions. Each part is self-contained and could be done independently. However, for Part III one should go over the likes of examples constructed in problems 3, 4 and 9 (you may do without geometric representation) from Part II for motivation. The first two parts are suitable for grade 4 and up. Part III could be done with grade 7 and up. Problems with an asterick * could be omitted without impacting the flow or understanding of the other problems.

0. In an honest, democratic vote process, who has the most power?

Who can influence the outcome most? Is it voters? Or those who campaign the best and can convince voters to change their mind?

Do you think it is possible with an honest, democratic voting procedure to pass a proposition or elect a candidate which *no one* wants?

We will see how those who make decisions about the voting procedure can manipulate the results - no cheating involved. In many examples they can get any outcome they want!

This math circle session is about voting paradoxes. What is a paradox? A contradiction. Something that is or seems logically impossible.

But first, a story.

Part I Voting and Logic

1. Problem 1. There was a kingdom once ruled by a king and a council of three members: Ana, Bob and Cory. It was a very democratic monarchy. It was the council which decided on all important issues by voting. The King could not even vote, but he was the only one who could bring propositions to vote. For example, he could call the council meeting in the morning and have them vote whether the King should have a nap after lunch.

One day, as King was coming up onstage to deliver his speech on the National Cherry Pie Day, he tripped and his crown fell off. In the awkward silence that followed, the King heard Count Olaf's laughter. His Majesty got very upset. He will punish the Count for this insulting laughter! He decided he will make Count Olaf watch the videos of the King's last three hundred speeches.

This important decision needed approval of the council. King privately talked to each member of the council only to find out that no one supports him. Ana was appalled by Count's behavior. However, when she heard of punishment, she paled and begged the King for mercy. "This is too cruel!" - said Ana. "Should Count Olaf be guilty of such behavior, he would have to be prosecuted with all strictness. However, I heard Olaf. He was just coughing. He is innocent!" - said Bob. Cory was of dangerous opinion that laughing at the King is not an insult, although he agreed that having to watch three hundred of King's speeches is an appropriate punishment for those who insult the King. In short, everyone was against the King's proposition. However, when the King brought the issue before the council to vote, Olaf's punishment was approved, with all members voting according to their opinions. What did the King do?

Helpful questions to students:

- what proposition the King should NOT put to vote?
- which propositions will be approved by majority?

Solution:

Each of the council member has own unique objection against the punishment of the Count. By questioning these objections one at a time King can have each of these overruled.

King put the following proposition to vote: "Laughing at the King is an insult".

How is this going to be voted on?

So, the law is passed: "Laughing at the King is an insult".

Another proposition: "Those who insult the King will be punished by having to watch the videos of the King's last 300 speeches."

Another proposition: "Count Olaf laughed at the King."

2. Table:

	Ana	Bob	Cory	majority
Laughing at the King is an insult	Y	Y	N	Y
Insult will be punished by making to watch 300 speeches	N	Y	Y	Y
Olaf laughed at the King	Y	N	Y	Y
Agree with all the above statements	N	N	N	N\Y

3. Why is this a paradox? Because one can have the council pass "No punishment for the count" or "Punishment for the count", depending which questions were asked. Thus we can obtain a contradiction. Can the king, then, get the council to pass ANY proposition? Such as:"The taxes will be increased by 200%"? We shall see that he can. We shall see how from a contradiction we could derive anything.

4. When constructing a mathematical model, we need to make some assumptions. These assumptions are usually not always true in reality. First, let us model the behavior of voters.

Rules for voters:

1. Always vote; you shall not abstain.
2. You shall not change your mind.
3. You shall be logically consistent .

This last rule will be explained in more detail later.

5. Let us practice the rules!

Mobile County Public School System is planning a radical school reform and YOU get to vote for the new rules for the schools.

Ask students for suggestions for the new rules. Here are some examples:
pajamas are the new school uniform;
cafeteria serves dessert only;
kids teach and adults are students;
teacher's speak in a rap;
video gaming is a required class with the homework, tests etc.;
every school has a spitball varsity team;

abolish 12th grade - you are done with school after 11th grade.

You (the teacher) may want to avoid rules which you anticipate will be unanimously approved or disapproved - you'll see why.

5.1 Let $A=$ "Pajamas are the new school uniform".
Have students vote on A. Those who vote "Yes" on A raise their hands. Denote the set of those voters by S_A. The set of all students is denoted by U. What can you tell about those who vote against A?

This is a good setting for the introduction of the basic logic and set operations, which I do not detail here. Logic studies propositions. Propositions are sentences which could be true or false. We can make new propositions out of those by using negation. Notation: for a proposition P, $\bar{P}$ means "not P".

What is the negation of A? Who votes "Yes" for the negation of A? Introduce complement $\bar{S}_A = U - S_A$.

Logical consistency Rule 1: if you vote "Yes" on proposition P then you vote "No" on $\bar{P}$ and vice versa.

5.2 Select another statement B. Have students vote on B. Then ask them to predict who will vote for the proposition "A and B".
$S_{A \text{ and } B} = S_A \cap S_B$

The solution to Problem 1 uses this rule - see the above table.

5.3 Similarly, for the proposition "A or B" we have $S_{A \text{ or } B} = S_A \cup S_B$.

Logical consistency for voters summary:
if you vote "Yes" on P, you vote against the negation of P;
vote "Yes" on proposition "A and B" exactly when you vote "Yes" on A AND "Yes" on B;
vote "Yes" to "A or B" when you vote "Yes" to A OR you vote "Yes" to B.

5.4* Optionally, you may have students verify experimentally that $S_{\overline{A \text{ and } B}} = \overline{S_A \cap S_B} = S_{\overline{A} \text{ or } \overline{B}}$.

6. If you (the teacher) were lucky, you had a situation where majority approved A and majority approved B but $S_A \cap S_B$ made up less than half of the students. Then you announce to students: "Now I will have you approve "No vacations!" proposition."

Put C= "There are no vacations" (i.e. school is year-round). Have students vote on the proposition "$\bar{A}$ or $\bar{B}$ or C". If they follow the Voter's Rules, they will approve this proposition. But A and B are already approved. It follows then that C is approved.

Of course, depending on the results you get you may have to replace A with $\bar{A}$ in the previous paragraph, or to replace B with $\bar{B}$. Keep track of the results of the votes and have students vote on various propositions until you get the situation you need. When I taught this for a large group of students, keeping track of votes was difficult. I called 3 volunteers to make the student council and we kept track of their votes. You need them to face away from each other, otherwise they tend to vote unanimously. Then getting a contradiction was quick.

7. When can I pass any proposition I want?

7.1 First requirement is to have a contradiction. This is possible when there are propositions A and B such that majority votes for A and a majority votes for B but "A and B" fails. Then we can pass A and we can pass B, therefore, we can pass "A and B"; but we can also fail "A and B".

When is it possible to get a contradiction?
When there is no majority which always votes the same way on all propositions.

7.2. Once you get this type of contradiction, to pass any proposition C have the council vote on "$\bar{A}$ or $\bar{B}$ or C". Have them vote on A and have them vote on B. All three propositions will be approved. From these three, it follows that C passed.

7.3. So, we (almost!) proved: Theorem (Shapiro, 1995): Given a council without a majority which is unanimous on all propositions, i.e. the council does not have a majority whose members vote the same on all issues, it is possible to pass any given proposition.

7.4 * This proof will not work in the case when $C = \overline{A}$. We will get an identically false statement approved by council. So, we need another approach. Problem 2* fixes this gap.

Problem 2. *. Suppose we have a council of three voters: Xavier (X), Yelena (Y) and Zane (Z). You know that the votes of X and Y agree on all propositions, except proposition A and, necessarily, some of it's logical derivatives, such as $\overline{A}$, "A and B " etc. X is against A while Y and Z are for it. Have this council approve $\overline{A}$.

Solution. We need to use A to create a contradiction. Let B be a proposition which is approved by both X and Y. Then it is only Y who votes for "A and B". Have the council vote on $\overline{\text{"}A \text{ and } B\text{"}}$. That will pass. Then have the council vote on B. This will pass, too. Thus we passed B, but not both A and B, therefore, A did not pass.

□

END OF PART I.

Historic Interlude
(see slides)

We will hear soon enough some strange words, such as "Borda" and "Condorcet". Until recently, it was considered that the subject, called "social choice theory", was pioneered by French mathematicians working in Paris in the late 18th Century, particularly, Jean-Charles DE BORDA and Marquis DE CONDORCET. However, in 2001 lost manuscripts of Ramon Llull (Catalan, Spain; c. 1232 - c. 1315) were found. LLull was a philosopher, logician, Franciscan monk. He is now credited with discovering the Borda count and Condorcet criterion, which Jean-Charles de Borda and Nicolas de Condorcet independently discovered centuries later. Also, Llull is recognized as a pioneer of computation theory, especially due to his great influence on Gottfried Leibniz.

One of the most celebrated and studied voting paradoxes is called "Arrow's Impossibility Theorem". We will learn about it in Part III. Kenneth Arrow, an economist, published the proof of his theorem in 1950. He received Noble's prize in Economics for his work.

Part II: Voting and Symmetry

Problem 3. a) Due to budget constraints, from now on, only one type of cookies will be served at Mobile Math Circle meetings. However, students are allowed to vote on the type of cookies to be served. Four choices were suggested (the choices in the handout are from our Math Circle paritcipants):

$$a,\ b,\ c,\ d$$

A poll was taken where each student ranked the four types of cookies in order of preference. Here are the results of the poll:

ranking\ number of students	5	3	5	4
1 st preference	a	a	b	c
2nd	d	d	c	d
3rd	c	b	d	b
4th	b	c	a	a

Such table is called a **voting profile**.

Def. A *voting profile* specifies the number of votes for each possible ranking.

What are possible ways to select a winner?

Here are some popular methods:

In **plurality method** the candidate with the most first-place votes (called the plurality winner) wins. Thus in plurality method, voters don't need to rank the candidates. The only information needed is the voters first choice.

Instant-runoff voting: Initially, only top choices are counted. Whoever is in last place, i.e. has least number of top choices, is eliminated from the race. Then the candidates ranked behind the eliminated candidate move up one place. The same method is repeated again.

Pairwise comparison, or the Condorcet criterion: If a candidate is preferred by the voters over each of the other candidates in a head-to-head comparison, then that candidate should be the winner of the election, called Condorcet winner.

The Borda Count Method: A candidate is given 3 points for each first place on the individual rankings, 2 points for the second place, 1 point for the 3rd place and 0 points for the last place. The candidate with the highest total sum of points is the winner.

Notation for ranking: $A \succ B$ means A ranks higher than B. A tie is denoted $A \sim B$.

Task: Find the winner as well as the complete ranking for each method.

a) Find the winner using the plurality method.

Answer:a

Complete ranking: $a \succ b \succ c \succ d$.

b) Find the winner using instant-runoff method.

Answer: b

Solution: We start by identifying the candidate in the last place. This is the same candidate that is in the last place for plurality method: d. We remove d from the race. Next, c is in the last place and we remove it. Then the vote is between a and b. 8 students will vote for a and 9 for b. Thus, the ranking is $b \succ a \succ c \succ d$.

c) Find the Condorcet winner.

Answer:c

d) Find the Borda Count winner.

Answer:d

Suppose your Math Circle organizer is secretly partial to butter cookies (choice a in the handout). She may say unassumingly: " Let us just have each person vote for the favorite cookie." (Plurality vote here.) Everyone will vote and think it is fair and reasonable, although most people prefer any other cookie to a. Notice, by suggesting a voting method she could make any of the four types of cookie win! However, in order to be able to do so, she needs to know the voting profile.

The above example used different voting methods. Let us look now at the scenario where inly one method is used.

Problem 4.

Let us introduce the **tournament method:** Put candidates names in some order. Vote between the first two candidates. The loser is eliminated and the winner goes on to compare with the third candidate, etc.

Consider the following scenario: Ann, Bob, Cory and Don are the candidates for a position in your class. 21 students will vote. After talking to these students I know their preferences for the candidates:

10 students: $A \succ B \succ C \succ D$

6 students: $B \prec C \prec D \prec A$

5 students: $C \prec D \prec A \prec B$

Tournament Method: select two candidates and vote. The loser is eliminated and the winner goes on to compare with the third candidate.

For example: Vote between A and B, for the profile above. A wins. Next, vote between A and C. C wins. Next, vote between C and D. C is the winner.

I organize the vote. Who would you like to win? For a small fee I can make any candidate win.

In fact, you can make this happen, too.

Who is an underdog in the voting profile?

a) Organize the tournament method so that Don wins.

b) A voting method satisfies the property of **unanimity** if whenever every voter ranks candidate X higher than candidate Y, the outcome of the vote should rank X higher than Y. Does the Tournament Method satisfy unanimity?

Note that everyone prefers C to D, yet D wins.

See slide: there is a cycle here.

Three candidate case.

Problem 5. In how many ways can one rank three candidates A, B, C (no ties allowed)?

Problem 6. *Geometric representation of a voting profile.* (Due to D.Saari)

Let us represent the three candidates A, B, C as the vertices of an equilateral triangle. An interior point P of this triangle represents a voter's preference based on the distance from P to the vertices: the one closer to P is ranked higher. If the distance to both vertices is the same then P represents a tie between the two candidates.

a) Find the locus of points representing $A \sim B$.

b) Find the set of all points within the triangle representing $A \succ B$.

c) Find the set of all points within the triangle representing $A \succ B \succ C$.

(Notice that if a vote is in the area $A \succ B$ and in the area $B \succ C$ then it is also in the area $A \succ C$.)

I put negations "not A", etc., on the sides of the triangle to indicate the last ranked candidate in the adjacent area. Thus, in the area adjacent to segment $A\overline{C}$, A is in the first place and C is in the last.

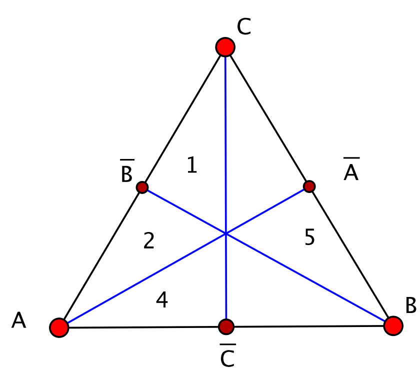
Example. Draw graphical representation using equilateral triangle for the voting profile:

4 voters: $A \succ B \succ C$

5 voters: $B \succ C \succ A$

1 voter: $C \succ A \succ B$

2 voters: $A \succ C \succ B$



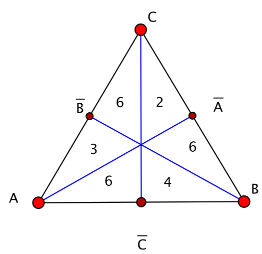
The slides demonstrate how tallies for plurality, pairwise and Borda could be calculated easily from geometric representation.

Problem 7. For the voting profile above write
a) pairwise tallies: the total number of $A \succ B$ votes under the segment $A\overline{C}$, etc. ;
b) the number of top choice votes by each vertex;
c) the Borda Count below this number.

Problem 8. Paradox: failure of positive association.

a) Use equilateral triangle to represent the following profile:

6 voters: $A \succ B \succ C$
4 voters: $B \succ A \succ C$
6 voters: $B \succ C \succ A$
2 voters: $C \succ B \succ A$
6 voter: $C \succ A \succ B$
3 voters: $A \succ C \succ B$



Do plurality ranking.

b) Using instant run-off method for this profile, who wins?

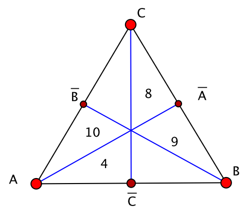
c) After a successful campaign by candidate A three voters changed their preferences from $B \succ A \succ C$ to $A \succ B \succ C$ and two voters changes their ranking from $C \succ B \succ A$ to $C \succ A \succ B$. Draw the new voting profile.

d) Who is the instant run-off winner now?

Problem 9. a) (non-transitivity, or Condorcet cycle) Construct a voting profile with 14 top choice votes for A, 8 top choice votes for B, 9 top choice for C and with pairwise ranking $A \succ B$, $B \succ C$ and $C \succ A$.
Hint: use representation triangle.

Talk to students about transitivity and non-transitivity. What is another non-transitive order that they know? Rock, paper, scissors.

b) (Reversal) Construct a voting profile with 14 top choice votes for A, 8 top choice votes for B, 9 for C (so that plurality ranking is $A \succ B \succ C$), but with Borda Count ranking $C \succ B \succ A$.



Problem.* Show that BC election tally is the sum of the pairwise tallies for the candidate.

Problem 10. Contribution of symmetries.

a) What are the profiles with the most symmetries?

Answer: see slide of Kernel.

What high symmetry profile means for candidates comparison?

- Every candidate is in the same position.

What should be the outcome of the election for such profile?

- A tie.

What are the results of plurality, Borda Count and Pairwise votes on such profile?

- Ties.

Examine how voting outcomes are affected by adding such profile to a given profile.

- They are not.

b) Find another profile where there is a symmetry between all three candidates. Answer: see slide of Condorcet profile.

c) Find the results of plurality, pairwise comparison and Borda count for a Condorcet profile.

d) How does addition of a Condorcet profile to an existing profile affects the results of plurality? Borda Count? Condorcet?

Reversal profile Another symmetric profile is Reversal. To obtain Reversal profile, start with one vote for a ranking, say, one vote for $A \succ B \succ C$. Then reverse this ranking, to obtain $C \succ B \succ A$ and add this vote to the voting profile. See slide. The idea is that the two votes should "cancel" each other. Does it happen? For which methods among Borda count, pairwise comparison and plurality this"cancellation" happens and there is a tie between candidates?

e) Find the results of plurality, pairwise comparison and Borda count for a Reversal profile.

f) How does addition of this profile to an existing profile affects the results of plurality? Borda Count? Condorcet?

Non-transitivity and connection to Part I

Consider propositions "$A \succ B$", "$B \succ C$". Transitivity means the truth of the statement "If $A \succ B$ and $B \succ C$ then $A \succ C$". Condorcet profile could be explained in the terms of the paradox which we learned in Part I:

	voter X	voter Y	voter Z	majority
$A \succ B$	T	F	T	T
$B \succ C$	T	T	F	T
$C \succ A$	F	T	T	T
Agree with all the above statements	F	F	F	F\T

Since we do not allow voters to have ties in their rankings, the statement "It is false that voter X ranks $C \succ A$" is equivalent to "X ranks $A \succ C$".

Problem 11. Subtract the largest Condorcet profile from the profile you constructed in Problem 9a). Use pairwise method to determine group ranking. Is there a cycle?

Problem 12. Subtract the largest Condorcet profile from the profile in Problem 8a). Solve the problem with the new profile. Is there a paradox?

Problem.* Subtract reversal profiles from the profile you constructed in problem 9b). Check the new profile to see if there is still a paradox.

Saari (1999): *Any discrepancies between the Borda Count ranking outcome and the pairwise outcome are due to a Condorcet component.*

Any discrepancies between the Borda Count outcome and the plurality outcome are due to Reversal components.

Therefore: to construct a profile with different Borda Count and pairwise rankings, for example, start by constructing a simple profile with the Borda count ranking outcome that you want and then add sufficiently large Condorcet Cycle.

To have a different plurality outcome, add sufficiently large reversal profile to those.

Problem 13.* a) Suppose in a three-candidate situation pairwise comparison results in a Condorcet cycle. Show that if we use a tournament method and start by comparing candidates X and Y and then between the winner and Z, then Z always wins.

b) If there is no Condorcet cycle then there is a Condorcet (pairwise comparison) winner who will always win the tournament procedure.

Part III Arrow's Theorem

In this part we shall continue to consider the case of three candidates.

Natural question: which voting procedure is the best? Kenneth Arrow, an economist, studied this question in the fifties. He approached it as a mathematician would.

Consider the set of all possible voting profiles. This is our domain, our inputs. For simplicity, we will allow only strict rankings by individual voters. A *voting system* is a function, i.e. a rule, which to every voting profile associates an output - a ranking of three candidates. We do allow ties in the output of a voting system.

Arrow asked a question: which of these functions satisfy some reasonable conditions that we want a fair voting system to satisfy? We will stay with the case of three voting alternatives although everything we do here could be applied to the situation with more voting alternatives.

Voting axioms:

Unanimity If every voter prefers the candidate X to the candidate Y then X will rank above Y in the outcome.

Which procedure did not satisfy unanimity? (Tournament)

Transitivity If X ranks above Y and Y ranks above Z in the outcome then X ranks above Z in the outcome. (Using short-hand notation: If $X \succ Y$ and $Y \succ Z$ then $X \succ Z$ in the outcome.) If X ties with Y and Y ties with Z in the outcome, then X ties with Z in the outcome. (Short-hand: If $X \sim Y$ and $Y \sim Z$ then $X \sim Z$.)

Which procedure did not satisfy transitivity? (Pairwise comparison)

There is one more axiom. To explain it, let us look at another paradox, perhaps least surprising and most familiar.

Problem 14. Suppose that in the scenario of Problem 2 it turned out that there are many chocolate chips cookies lovers and also a significant number of oatmeal raisin cookie lovers. As a result, three candidates emerge with the votes for the top choice split

Keebler Chips Deluxe 32%, Chips Ahoy! 31%, Oatmeal Raisin 37%.

a) Which type of cookies is the plurality winner?

b) Suppose that Keebler Chips Deluxe is discontinued. Then the vote is between the two remaining types of cookies. Which type is the winner now?

Answer: the assumption is that all those who voted for Keebler will now vote for Chips Ahoy! which will be the winner then. This is a familiar effect of third-party spoiler in presidential elections.

Independence of Irrelevant Alternatives (IIA): removal of a candidate should not affect the relative ranking of the other two candidates in the outcome of the vote. That is, the ranking of the candidates X and Y by the voting system depends only on the ranking of X and Y by voters and does not depend on rankings of Z in the voting profile.

c) Does the plurality method satisfy the Independence of Irrelevant Alternatives property?

Arrow set a task for himself: find voting systems which satisfy transitivity, unanimity and IIA.

Let us follow Arrow in figuring it out.

But first, let us estimate our chances of success in finding such functions.

If a function satisfies IIA, its output can be calculated like this: we first look at $A : B$ rankings turning blind eye to any rankings of C. This information will decide the $A : B$ ranking in the outcome. Similarly, we independently decide on $B : C$ ranking in the outcome and $A : C$ ranking. Then we put all three rankings together and pray for transitivity. From what we already know, our chances of success are not great. Our task is to find a rule which will guarantees success for every voting profile!

Does such rule exist? It seems unlikely. IIA is a very strong condition. Transitivity implies that pairwise rankings are not independent of each other. It turns out, there are rules which satisfy the above axioms but they are not what anyone would hope for...

The proof outlined in the exercises is not the original Arrow's proof. It is a combination of proofs by Sridhar Ramesh and by Terrence Tao. There are many proofs of Arrow theorem. They are elementary (or could be translated into elementary language), but a bit too long and tedious to be done in a math circle. However, the proof below is relatively short, transparent and can be done in a series of easy engaging exercises, appropriate for grades 7 and up.

First, let us introduce the notion of a winning set for a voting system. Suppose that for a particular profile $A \succ B$ in the outcome of the the voting system. According to IIA this depends only on $A : B$ ranking of voters. We say that the set M of all voters who ranked A above B in this profile *wins for A over B*(for $A \succ B$). Notice that all voters NOT in M, i.e. those in $\overline{M}$, by definition, rank $B \succ A$ in this particular profile.

In the remaining problems we assume that axioms of transitivity, unanimity and IIA are satisfied.

Problem 15. a) Does the set of all voters wins for a candidate over another? Why?
Answer:Yes, because of unanimity.

b) Can it be that an empty set wins for A over B?
Answer: No, since then the unanimity is violated.

Problem 16. *Set that wins for one wins for all.*

a) Show that if a set M wins for $A \succ B$ then it wins for $A \succ C$.

Note on pedagogy: this is a statement for older students. An opportunity to introduce them to proofs. As almost with any proof, the first thing is to ask: what is given? what do we need to show?

The immediate answers to these are: Given: M wins for $A \succ B$. Show: M wins for $A \succ C$. As a rule this first answer is not too helpful. We need to be more specific. In this case, as it is often the case, we need to unravel the meaning of the words, the definitions. So, more precise version is:

Given: M is in the left half of the triangle, $\overline{M}$ in the right half.

Task: construct a profile which satisfies the above conditions; in addition, M is in $A \succ C$ area, $\overline{M}$ is in $C \succ A$, and which guarantees the outcome $A \succ C$.

For the younger students, here is another version of this problem:

Problem 16 Suppose M wins for $A \succ B$. Consider a profile where

voters in M all vote $A \succ B \succ C$

voters not in M vote $B \succ C \succ A$.

What can you say about $A : B$ ranking in the outcome? $B : C$ ranking? $A : C$? What is the winning set for $A \succ C$?

b) Similarly, show that if a set N wins for $B \succ A$ then it wins for $C \succ A$.

Hint: reverse the ratings in the proof of part a).

c) Show that if a set wins for $A \succ B$ then it wins for all 6 possible ratings of pairs.

Solution. Suppose a set wins for A over B, then by a) it also wins for A over C, therefore, by b) also for B over C, then, by a) for B over A. Also, by b) for C over B and then by a) for C over A.

Thus, it makes sense to talk about a winning set.

Problem 17. *No ties.*

Show that there cannot be ties in the outcome of voting.

Hint: assume that for some voting profile the outcome is $A \sim B$. Consider the set M of all voters who rank $A \succ B$. Use the same voting profile as in the previous problem to arrive to contradiction.

Solution: Why can't M be a winning set for any pair? What are possible outcomes of this vote?

Using the same profile as in problem 16 above, outcome of the vote must be $A \sim B$,$B \succ C$ and either 1°. $A \sim C$ or 2°. $C \succ A$.
In case 1° transitivity is violated. In case 2°, $\overline{M}$ is a winning set for $C \succ A$ and , therefore for $B \succ A$, so then we can't have $A \sim B$ in the outcome. □

Problem 18. Corollary. *For any set of voters M either M or it's complement $\overline{M}$ is a winning set.* Proof: Suppose voters in M all vote $A \succ B$ and voters in $\overline{M}$ all vote $B \succ A$. In the outcome either one of this rankings must hold.

Problem 19. *Intersection of winning sets is a winning set.*

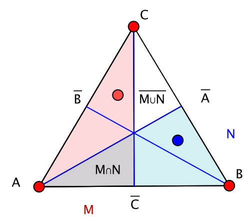
Let M and N be two winning sets. Consider a profile where voters in M rank $A \succ B$, voters in $\overline{M}$ rank $B \succ A$; voters in N rank $B \succ C$, voters in $\overline{N}$ vote $C \succ B$.

a) What can you say about the $A : C$ ranking in the outcome?

Answer: $A \succ C$ by transitivity.

b) Construct such profile with an additional condition that only voters in $M \cap N$ rank $A \succ C$. Hint: use representation triangle.

Solution:



□

Problem 20. Corollary *There are no disjoint winning sets.*

Problem 21. There exists a *dictator*, i.e. a distinguished voter v so that $\{v\}$ is a winning set and all other winning sets are exactly the sets which contain v.

Proof. Pick any voter v_1.

Suppose $\{v_1\}$ is a winning set. Then by the previous Corollary a set which does not contain v_1 cannot be winning. On the other hand, any set which contains v_1 must be winning, otherwise its complement, which does not contain v_1, would be winning. We are done.

If $\{v_1\}$ is not a winning set then its complement U_1 must be a winning set. It is non-empty. If it has just one voter, then we are done. Otherwise pick a voter $v_2 \in U_1$. If $\{v_2\}$ is a winning set, we are done. Otherwise $U_2 = U - \{v_2\}$ is a winning set, therefore $U_1 \cap U_2 = U - \{v_1, v_2\}$ is a winning set. Continuing this way guarantees that we will get a one-voter winning set. Pedagogy: to make it visual do a hands-on demonstration with the students, picking one student at a time. □

Problem 22. The voting system in the previous problem is the same as the following: there is a voter v such that the voting outcome coincides with the ranking given by v ignoring the rankings of other voters. Such system is called *a dictatorship.*

Arrow's Theorem. If a voting system with three or more candidates satisfies unanimity, transitivity and IIA then it is a dictatorship.

Problem 23.* In the original statement of the theorem Arrow allows ties in the voting profile. Explain why the conclusion of the theorem still holds true for this case. (Do this for three candidates).

Solution: First, check directly that dictatorship still satisfies the three axioms when ties are allowed in the voting profile.

Secondly, we need to understand why there can't be any other voting procedures satisfying the axioms. By allowing ties we expanded our domain. A set of voting profiles without ties is now a subset of this domain. Check that if a function satisfies each of the three axioms then this still should hold true on a subset of the domain. But we already proved that on this subset it is only a dictatorship that is possible. □

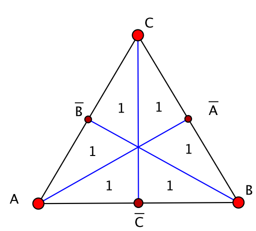
Arrow's theorem has a dramatic, if not sinister, ring to it due to the appropriate name "dictatorship" of a voting system function. Of course, the name "dictator" has nothing to do with moral qualities of the voter in this abstract model. A function of several variables, whose output coincides with the value of one chosen variable, thus ignoring the values of other variables, is called a *projection map.* These simple functions are used a lot in mathematics. In our model of voting, the number of variables is the number of voters, each variable can take one of the six possible values (rankings of the three candidates) and "dictatorship" is the name of a projection map. Here is a less-dramatic version of the theorem:

Arrow impossibility theorem. A voting system with three or more candidates which satisfies unanimity, transitivity, IIA and is not a dictatorship, does not exist.

Math Cheat for teachers

A voting profile for the ranking of three candidates is a point in a six dimensional space. D. Saari [Saari, 1999] introduced four pairwise orthogonal subspaces which span this six-dimensional space.

One of them is one dimensional Kernel subspace spanned by



A *profile differential* is the difference between two profiles involving the same number of voters.

The other three subspaces consist of profile differentials (see slides):

two dimensional Basic subspace,
one dimensional Condorcet subspace,
two dimensional Reversal subspace.

Theorem [Saari, 1999]

1. On the Basic subspace plurality, Borda Count and pairwise rankings agree.

2. Adding a non-zero element of the Condorcet subspace does not change plurality or Borda Count outcomes, but adds a cycle to the pairwise ranking.

3. Adding a non-zero reversal component does not change Borda Count or pairwise ranking, but changes plurality tallies.

4. (”Arrow’s Possibility Theorem”) Consider the set of all voting profiles with no Condorcet component (i.e. the five dimensional subspace orthogonal to Condorcet subspace). Voting system functions on this set which satisfy transitivity, unanimity and IIA include Borda Count, pairwise ranking and some other methods.

References:

Part I: *Voting and Logic* Shapiro’s Theorem is in A.Shapiro, *Logic and Parliament*(1995), Kvant, 1995 ,03 (in Russian).

Problem 1 is a version of so-called ”Doctrinal Paradox” or ”Discursive Dilemma”.

Part II: *Voting and Symmetry* is based on some results from

D.G. Saari, *Explaining all three-alternative voting outcomes*, Journal of Economic Theory 87, 313 - 355 (1999)

Part III: Arrow’s Theorem The proof is an amalgam of the proofs of the Arrow’s Theorem by Sridhar Ramesh (https://pleasantfeeling.wordpress.com/2009/04/19/arrowstheorem/) and by Terrence Tao (https://www.math.ucla.edu/ tao/arrow.pdf)