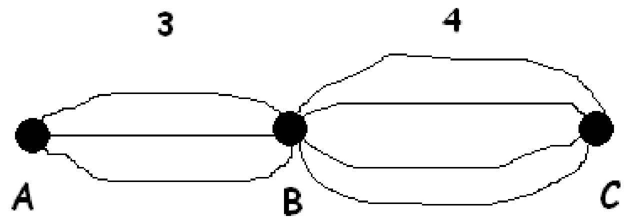
CHAPTER SIXTEEN

☆ COUNTING PRINCIPLES ☆

THE MULTIPLICATION PRINCIPLE

Here's a very simple puzzle:

There are three major highways from Adelaide to Brisbane, and four major highways from Brisbane to Canberra.



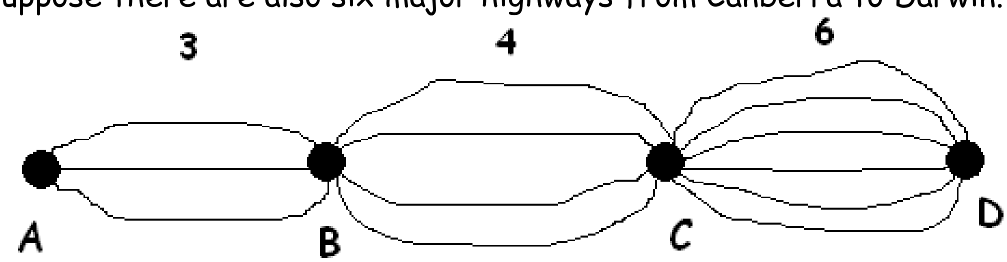
How many different routes can one take to travel from Adelaide to Canberra?

The answer to this question is clearly 12. But pause for a moment and ask yourself why? Is it obvious that the number of routes from A to C really is 3×4, that is, three groups of four?

Make sure you are comfortable that "multiplication" really is the right arithmetic operation here (as opposed to direct addition).

Let's take the puzzle up a notch:

Suppose there are also six major highways from Canberra to Darwin.



How many different routes are there from A to D?

Be sure that you are convinced the answer is given by multiplication:

#routes = $3 \times 4 \times 6 = 72$

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

EXERCISE: I own five different shirts, four different pairs of trousers and two sets of shoes. How many different outfits could you see me in?

EXERCISE: There are ten possible movies I can see and ten possible snacks I can eat whilst at the movies. I am going to see a film tonight and I will eat a snack. How many choices do I have in all for a movie/snack combo?

We have the …

THE MULTIPLICATION PRINCIPLE
If there are *a* ways to complete one task and *b* ways to complete a second task, and the outcomes of the first task in no way affect the choices made for the second task, then the number of different ways to complete both tasks is $a \times b$.

This principle readily extends to the completion of more than one task.

EXERCISE: Explain the clause stated in the middle of the multiplication principle. What could happen if different outcomes from the first task affect choices available for the second task? Give a concrete example.

FACTORIALS:

In how many ways can six people stand in a line?

<u>Answer</u>: There are six possibilities for the task of placing someone in the first spot, five possibilities for who to place second, four for third, three for fourth, two for the fifth and one for sixth. By the multiplication there are thus:

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

ways to complete the task of lining up all six people. □

Definition: The product of integers from 1 to N is called N *factorial* and is denoted N!.

These numbers grow very large very quickly:

$$1! = 1$$
$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$
$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$
$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$
$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

COMMENT: In 1729, at the age of 22, Swiss mathematician Leonhard Euler found a formula for a function that generalizes the factorial function. He called it the *gamma function*. The curious thing is that one can input fractional and irrational values into his gamma function and obtain meaningful answers. Euler discovered, for instance, that $(\frac{1}{2})!$ equals $\frac{\sqrt{\pi}}{2}$.
Very strange!

EXERCISE: What is the highest factorial your calculator can handle?

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

WORD GAMES:

EXAMPLE: My name is JIM. In how many ways can one rearrange the letters of my name?

Answer 1: By brute force we can list all possibilities and see that there are six arrangements: JIM JMI MIJ MJI IMJ IJM

Answer 2: We can use the multiplication principle. We have three slots to fill:

___ ___ ___

The first task is to fill the first slot with a letter. There are 3 ways to complete this task.

The second task is to fill the second slot. There are 2 ways to complete this task. (Once the first slot is filled, there are only two choices of letters to use for the second slot.)

The third task is to fill the third slot. There is only 1 way to complete this task (once slots one and two are filled).

3 2 1

By the multiplication principle, there are thus $3 \times 2 \times 1 = 3!$ ways to complete this task. □

EXERCISE: In how many ways can one arrange the letters HOUSE ?

EXERCISE: How many ways are there to rearrange the letters BOB? Assume the Bs are indistinguishable?

Comment: One can certainly answer this second exercise by brute force – just list the possibilities. But is there a sophisticated way to think about how to handle the repeated letter? Think about this before reading on.

PRACTICE EXERCISE:
In how many ways can one arrange the letters HOUSES?

Certainly if the Ss were distinguishable – written, say, as S_1 and S_2 - then the problem is easy to answer:

There are 6! ways to rearrange the letters $HOUS_1ES_2$.

The list of arrangements might begin:

$HOUS_1ES_2$
$HOUS_2ES_1$

$OHUS_1S_2E$
$OHUS_2S_1E$

S_1S_2UEOH
S_2S_1UEOH
$\vdots$

But notice, if the Ss are no longer distinguishable, then pairs in this list of answers "collapse" to give the same arrangement. We must alter our answer by a factor of two and so the number of arrangements of the word HOUSES is:

$$\frac{6!}{2} = 360$$

<u>QUESTION</u>: What is this "2" on the denominator? To properly understand it, work out the answer to this next problem:

EXERCISE: How many ways are there to rearrange the letters of the word CHEESE?

Think about this before reading on.

Answer: If the three Es are distinct – written E_1, E_2, and E_3, say – then there are 6! ways to rearrange the letters $CHE_1E_2SE_3$. But the three Es can be rearranged 3! = 6 different ways within any one particular arrangement of letters. These six arrangements would be seen as the same if the Es were no longer distinct:

$$\begin{matrix} HE_1E_2SCE_3 & HE_3E_1SCE_2 & & \\ HE_1E_3SCE_2 & HE_3E_2SCE_1 & \rightarrow & HEESCE \\ HE_2E_1SCE_3 & HE_2E_3SCE_1 & & \end{matrix}$$

Thus we must divide our answer of 6! by 3! to account for the groupings of six that become identical. There are thus $\frac{6!}{3!} = 120$ ways to arrange the letters of CHEESE.

□

Comment: The number of ways to rearrange the letters HOUSES is $\frac{6!}{2!}$. The "2" on the denominator is really 2!.

EXERCISE: Explain why the number of ways to arrange the letters of the word CHEESES is $\frac{7!}{3!2!}$.

EXERCISE:
In how many ways can one arrange the letters CHEEEEESIEST?
How about of CHEESIESTESSNESS?

<u>Comment</u>: Consider the word DOODLED. Its letters can be arranged $\frac{7!}{2!3!}$ different ways, with 2! in the denominator arising from the fact that there are two Os, and the 3! from the three Ds. If we wished, we could also include in the denominator a 1! (which equals 1) for the fact that there is a single L in the word and another 1! for the single E. Thus the number of ways to arrange the letters DOODLED might be better written:

$$\frac{7!}{2!3!1!1!}$$

This has the advantage of offering a "self check:" the numbers in the denominator should match - in sum - the numbers in the numerator.

Let's take this further …

Each number in the denominator corresponds to the number of times a letter appears in the original word: O two times, D thrice, E once and L once. The letter P appears zero times so we could actually write:

$$\frac{7!}{2!3!1!1!0!}$$

Also, the letter J appears zero times as well, so perhaps we should write:

$$\frac{7!}{2!3!1!1!0!0!}$$

and so on.

THIS IS ALL FINE AND CONSISTENT IF WE CHOOSE TO DEFINE 0! TO BE THE NUMBER 1.

It is for this reason that mathematicians set 0! = 1. Even if one is being silly, the formulas remain correct.

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

THE LABELING PRINCIPLE

We can rephrase the letter-arranging problem. Again consider the word CHEESIEST. Rearranging these letters corresponds to assigning letters to nine slots:

___ ___ ___ ___ ___ ___ ___ ___ ___

1 slot is to be "labeled" C
1 slot is to be labeled H
3 slots are to be labeled E
2 slots are to be labeled S
1 slot is to be labeled I
1 slot is to be labeled T

We know the answer to the problem is: $\frac{9!}{1!1!3!2!1!1!}$.

This is the same problem as the following:

> Nine people are to be given hats. One is to be given a cranberry-red hat (C), one is to be given a hot-pink hat (H), three emerald-green hats (E), two sky-blue hats (S), one an indigo-blue hat (I), and one a teal hat (T). How many ways?

We see that rearranging letters is equivalent to assigning labels to distinct objects (people or specific slots) and the answer to the problem is the fraction with numerator the number of objects, factorialised, and denominator given by the counts of objects with each label, factorialised.

We have:

THE LABELING PRINCIPLE
Each of distinct N objects is to be given a label. If k_1 of them are to have label "1," k_2 label "2," and so on, all the way to k_r of them label "r," then total number of ways to assign all labels is given by:

$$\frac{N!}{k_1!k_2!\cdots k_r!}$$

This is an extremely powerful result.

SOME EXAMPLES:

1. *Four people from a group of ten are needed for a committee. In how many different ways can a committee be formed?*

Answer: The ten folk are to be labeled as follows: 4 as "on the committee" and 6 as "off." The answer must be $\frac{10!}{4!\,6!}$. □

COMMENT: Notice that we were sure to assign an appropriate label to each and every person (or object) in the problem.

2. *Fifteen horses run a race. How many possibilities are there for first, second, and third place?*

Answer: One horse will be labeled "first," one will be labeled "second," one "third," and twelve will be labeled "losers." The answer must be: $\frac{15!}{1!\,1!1!12!}$. □

3. *A "feel good" running race has 20 participants. Three will be deemed equal "first place winners," five will be deemed "equal second place winners," and the rest will be deemed "equal third place winners." How many different outcomes can occur?*

Answer: Easy! $\frac{20!}{3!\,5!12!}$. □

4. *From an office of 20 people, two committees are needed. The first committee shall have 7 members, one of which shall be the chair and 1 the treasurer. The second committee shall have 8 members. This committee will have 3 co-chairs and 2 co-secretaries and 1 treasurer. In how many ways can this be done?*
Answer: Keep track of the labels. Here they are:

1 person will be labeled "chair of first committee"
1 person will be labeled "treasure of first committee"
5 people will be labeled "ordinary members of first committee"
3 people will be labeled "co-chairs of second committee"
2 people will be labeled "co-secretaries of second committee"
1 person will be labeled "treasurer of second committee"
2 people will be labeled "ordinary members of the second committee"
5 people will be labeled "lucky," they are on neither committee.

The total number of possibilities is thus: $\frac{20!}{1!1!5!3!2!1!2!5!}$. Easy! □

COMMENT: Students are usually taught to distinguish between a "permutation" and a "combination." They differ by whether or not the order of terms is important. This is unnecessarily confusing – and somewhat artificial. People would call the first example a combination. They would call the second example a permutation. There are no names for examples 3 and 4!

The formula $\frac{N!}{k_1!k_2!\cdots k_r!}$ with $k_1 + k_2 + \cdots + k_r = N$ is called a *generalized combinatorial coefficient*. It is denoted:

$$\begin{pmatrix} N \\ k_1 \ k_2 \ \cdots \ k_r \end{pmatrix}$$

For example, $\begin{pmatrix} 6 \\ 2\ 3\ 1 \end{pmatrix} = \frac{6!}{2!3!1!} = 60$

EXAMPLE: *In how many different ways can one arrange seven As and nine Bs?*

Answer: We have sixteen "slots," seven of which are to be labeled "A" and nine to be labeled "B." This gives:

$$\frac{16!}{7!9!}$$

possible arrangements.

EXAMPLE: *Ten circles are drawn in a row. In how many different ways can we color two of them black and leave the rest white?*

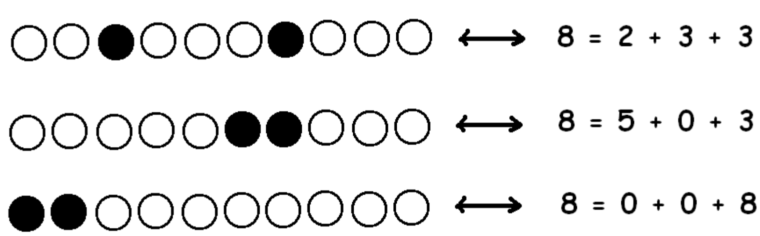
Answer: Two circles are to be "labeled" black and eight as white. There are:

$$\frac{10!}{2!8!} = \frac{10\times 9}{2} = 45$$

possibilities.

CHALLENGE: *How many solutions are there to the equation* $8 = a + b + c$ *if each of* a, b *and* c *is a positive integer or zero?*

<u>HINT</u>:



IF YOU REALLY ARE WORRIED ABOUT ORDER …

1. "SELECTION WITHOUT ORDER" IS JUST LABELING

An example will explain:

Suppose 5 people are to be chosen from 12, and the order in which folk are chosen is not important. In how many ways can this be done?

<u>Answer</u>:

5 people will be labeled "chosen" and 7 "not chosen. There are $\frac{12!}{5!7!}$ ways to accomplish this task. □

2. "SELECTION WITH ORDER" IS JUST LABELING

An example will again explain:

Suppose 5 people are to be chosen from 12 for a team and the order in which they are chosen is considered important. In how many ways can this be done?

<u>Answer</u>: We have:

1 person labeled "first"
1 person labeled "second"
1 person labeled "third"
1 person labeled "fourth"
1 person labeled "fifth"
7 people labeled "not chosen"

This can be done $\frac{12!}{1!1!1!1!1!7!}$ ways. □

Again … there is no need to fuss about order. Just come up with the labeling scheme that is appropriate for the problem.

EXERCISE: Coming full circle … Explain, using the labeling principle, why the number of ways to arrange six people in a line is 6! (which is really $\frac{6!}{1!1!1!1!1!1!}$)

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

MULTI-STAGE LABELING

Although the labeling principle helps remove the confusion of order vs. non-order, many standard "arrangement" problems still possess a level of complication that is delicate. For example, consider the following typical standardized test problem:

> *In how many ways can one arrange the letters of the word ORANGE if the first and last letters must each be a vowel?*

This is not a straightforward labeling problem as some objects are given preferred status over others: the vowels require a different type of consideration from the consonants. It is really a two-stage challenge:

> STAGE 1: Contend with the vowels
> STAGE 2: Contend with the remaining letters

Each of these stages can be handled separately. The Multiplication Principle tells us to then multiply the results.

Solution:
STAGE 1: One vowel shall be labeled "first position," one "last position" and one shall be labeled "placed with the consonants." There are $\frac{3!}{1!1!1!} = 6$ ways to complete stage one.

STAGE 2: We now have four "consonants," R, N, G, and the remaining vowel, to label as second, third, fourth and fifth. There are $\frac{4!}{1!1!1!1!} = 24$ ways to accomplish stage two.

Thus there are $6 \times 24 = 144$ desired arrangements of ORANGE. □

COMMENT: Many might prefer to present the answer as a six-stage process:

$$\underline{\;3\;}\quad\underline{\;4\;}\quad\underline{\;3\;}\quad\underline{\;2\;}\quad\underline{\;1\;}\quad\underline{\;2\;}\quad = 144$$

Do you see what is meant by this diagram?

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

EXAMPLE: A company would like to send out a team of five plumbers to a construction site. They will send two expert plumbers and three trainee plumbers. If there are a total of 10 expert plumbers available and 8 trainees, how many different teams are possible?

Answer: This too is a two stage process:

STAGE 1: Select the experts

There are $\frac{10!}{2!8!}$ possible ways to label two expert plumbers as "chosen" and the rest "not chosen."

STAGE 2: Select the trainees

There are $\frac{8!}{3!5!}$ possible ways to label three trainees as "chosen" and the rest "not chosen."

By the multiplication principle, there are thus $\frac{10!}{2!8!} \times \frac{8!}{3!5!}$ possible teams. □

COMMENT: Notice that we have no control over who is labeled "expert" and who is labeled "trainee." We only have control over the labels "chosen" and "not chosen." That there some fixed previously assigned labels is a hint that this must be dealt as a multi-stage problem.

EXAMPLE: In how many ways can one arrange the letters ABCDE so that A is never at the beginning or the end?

We'll give three answers to this problem, even though most people would prefer to answer the question just the first we present. (We offer two more approaches just to illustrate that there are multiple ways to approach these problems.)

Answer 1: Think of this as a five-stage process! Deal with the first letter, deal with the last letter, deal with the second letter, deal with the third letter, and deal with the fourth letter. By the multiplication principle, we multiply the results.

$$\underline{\;4\;}\quad\underline{\;3\;}\quad\underline{\;2\;}\quad\underline{\;1\;}\quad\underline{\;3\;}\quad = \quad 72$$

Answer 2: The letters B, C, D, E have a different status than A.

STAGE 1: Place the letter A

There are 3 possible locations for this letter

STAGE 2: Place the remaining letters

There are four remaining positions for four letters. They can be placed in these positions $\frac{4!}{1!1!1!1!} = 24$ ways.

Thus there are $3 \times 24 = 72$ desired arrangements.

Answer 3: There are five slots in which to place letters with the two end slots having a different status than the middle three.

STAGE 1: Fill the end slots.

There are four letters to work with, yielding $\frac{4!}{1!1!2!} = 12$ possibilities.

(The labels here are "first slot," "last slot" and "not used.)

STAGE 2: Fill the middle three slots

There are three letters to work with yielding $\frac{3!}{1!1!1!} = 6$ possibilities.

By the multiplication principle we have $12 \times 6 = 72$ permissible arrangements. □

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

EXAMPLE: Six people Albert, Bilbert, Cuthbert, Dilbert, Egbert and Filbert are to sit in a circle. How many different arrangements are possible if rotations of the same arrangement are considered equivalent?

Answer: This question is tricky in that there are no clear "labels" associated with the question: there is no clear "first" seat or "second" seat, and so forth. We can think of it as a multi-stage process nonetheless by having the men take a seat one at a time:

> Albert must sit somewhere. He can sit anywhere (since all rotations are deemed equivalent) and there is thus only 1 action for him to take.
>
> Bilbert now has 5 options: take the seat one place to Albert's left, two places to his left, and so on. Cuthbert has 4 options. Dilbert has 3. Egbert has 2. Filbert has 1.
>
> Thus by the multiplication principle, there are $1\times 5\times 4\times 3\times 2\times 1 = 120$ possible arrangements.

We can be a little slicker and think of this as a two-stage process:

STAGE 1: Albert takes a seat

> There is only 1 option: Albert takes any seat.

STAGE 2: The remaining five each take a seat.

> This is a labeling problem as Albert's position now defines five labels: one place to his left, two places to his left, and so on. There are thus $\frac{5!}{1!1!1!1!1!} = 120$ possibilities.

By the multiplication problem there are $1\times 120 = 120$ possible configurations.

□

FUN WITH POKER HANDS

One plays poker with a deck of 52 cards, which come in 4 suits (hearts, clubs, spades, diamonds) with 13 values per suit (A, 2, 3, …, 10, J, Q, K).

In poker one is dealt five cards and certain combinations of cards are deemed valuable. For example, a "four of a kind" consists of four cards of the same value and a fifth card of arbitrary value. A "full house" is a set of three cards of one value and two cards of a second value. A "flush" is a set of five cards of the same suit. The order in which one holds the cards in ones hand is immaterial.

EXAMPLE: How many flushes are possible in poker?

Answer: Again this is a multi-stage problem with each stage being its own separate labeling problem. One way to help tease apart stages is to image that you've been given the task of writing a computer program to create poker hands. How will you instruct the computer to create a flush?

First of all, there are four suits – hearts, spades, clubs and diamonds – and we need to choose one to use for our flush. That is, we need to label one suit as "used" and three suits as "not used." There are $\frac{4!}{1!3!} = 4$ ways to do this.

Second stage: Now that we have a suit, we need to choose five cards from the 13 cards of that suit to use for our hand. Again, this is a labeling problem - label five cards as "used" and eight cards as "not used." There are $\frac{13!}{5!8!} = 1287$ ways to do this.

By the multiplication principle there are $4 \times 1287 = 5148$ ways to compete both stages. That is, there are 5148 possible flushes. □

Comment: There are $\frac{52!}{5!47!} = 2598960$ five-card hands in total in poker. (Why?) The chances of being dealt a flush are thus: $\frac{5148}{2598960} \approx 0.20\%$.

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

EXAMPLE: How many full houses are possible in poker?

Answer: This problem is really a three-stage labeling issue.

First we must select which of the thirteen card values – A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K - is going to be used for the triple, which will be used for the double, and which 11 values are going to be ignored. There are $\frac{13!}{1!1!11!} = 13 \times 12 = 156$ ways to accomplish this task.

Among the four cards of the value selected for the triple, three will be used for the triple and one will be ignored. There are $\frac{4!}{3!1!} = 4$ ways to accomplish this task. Among the four cards of the value selected for the double, two will be used and two will be ignored. There are $\frac{4!}{2!2!} = 6$ ways to accomplish this.

By the multiplication principle, there are $156 \times 4 \times 6 = 3744$ possible full houses. □

COMMENT: High-school teacher Sam Miskin recently used this labeling method to count poker hands with his high-school students. To count how many "one pair hands" (that is, hands with one pair of cards the same numerical value and three remaining cards each of different value) he found it instructive bring 13 students to the front of the room and hand each student four cards of one suit from a single deck of cards.

He then asked the remaining students to select which of the thirteen students should be the "pair" and which three should be the "singles." He had the remaining nine students return to their seats.

He then asked the "pair" student to raise his four cards in the air and asked the seated students to select which two of the four should be used for the pair. He then asked each of the three "single" students in turn to hold up their cards while the seated students selected on one the four cards to make a singleton.

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

This process made the multi-stage procedure clear to all and the count of possible one pair hands, namely,

$$\frac{13!}{1!3!9!} \times \frac{4!}{2!2!} \times 4 \times 4 \times 4$$

readily apparent.

EXERCISE: "Two pair" consists of two cards of one value, two cards of a different value, and a third card of a third value. What are the chances of being dealt two-pair in poker?

EXAMPLE: A "straight" consists of five cards with values forming a string of five consecutive values (with no "wrap around"). For example, 45678, A2345 and 10JQKA are considered straights, but KQA23 is not. (Suits are immaterial for straights.)

How many different straights are there in poker?

Answer: A straight can begin with A, 2, 3, 4, 5, 6, 7, 8, 9 or 10. We must first select which of these values is to be the start of our straight. There are 10 choices.

For the starting value we must select which of the four suits it will be. There are 4 choices.

There are also 4 choices for the suit of the second card in the straight, 4 for the third, 4 for the fourth, and 4 for the fifth.

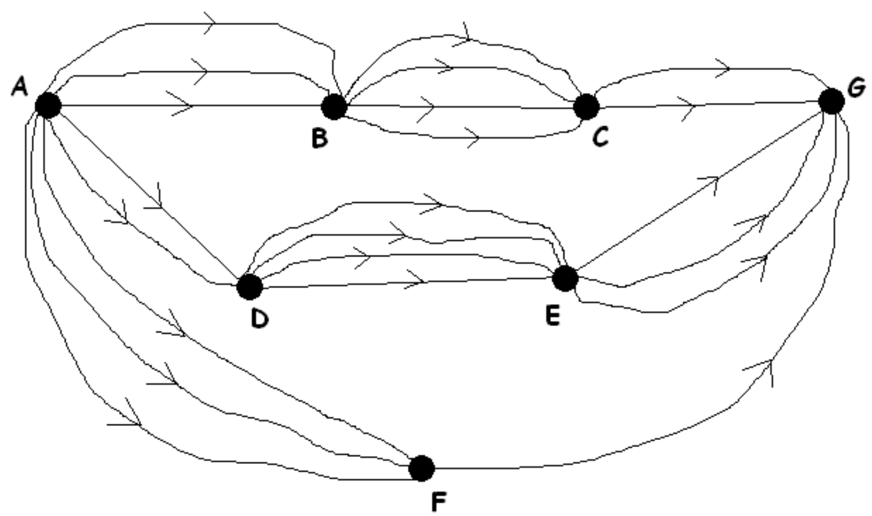
By the multiplication principle, the total number of straights is:

$$10 \times 4 \times 4 \times 4 \times 4 \times 4 = 10240.$$

The chances of being dealt a straight is about 0.39%. □

EXERCISES

Question 1: How many different paths are there from A to G?



Question 2:

a) In how many different ways can one arrange five As and five Bs.
b) A coin is tossed 10 times. In how many different ways could exactly five heads appear?

Question 3: The word BOOKKEEPING is the only word in the English language with three consecutive double letters. In how many ways can one arrange the letters of this word?

Question 4: In how many ways can you arrange the letters of your full name?

Question 5:

a) A mathematics department has 10 members. Four members are to be selected for a committee. In how many different ways can this be done?
b) A physics department has 10 members and a committee of four is needed. In that committee, one person is to be selected as "chair." In how many different ways can one form a committee of four with one chair?
c) An arts department has 10 members and a committee of four is needed. This committee requires two co-chairs. In how many different ways can one form a committee of four with two co-chairs?
d) An English department has 10 members and two committees are needed: One with four members with two co-chairs and one with three members and a single chair. In how many different ways can this be done?

Question 6: Evaluate the following expressions:

a) $\frac{800!}{799!}$ b) $\frac{15!}{13!2!0!}$ c) $\frac{87!}{89!}$

Simplify the following expressions as far as possible:

d) $\frac{N!}{N!}$ e) $\frac{N!}{(N-1)!}$ f) $\frac{n!}{(n-2)!}$

g) $\frac{1}{k+1} \cdot \frac{(k+2)!}{k!}$ h) $\frac{n!(n-2)!}{((n-1)!)^2}$

Question 7: Three people from 10 will be asked to sit on a bench: one on the left end, one in the middle, and one on the right end. In how many different ways can this be done?

Question 8: Five pink marbles, two red marbles, and three rose marbles are to be arranged in a row. If marbles of the same colour are identical, in how many different ways can these marbles be arranged?

Question 9:

a) Hats are to be distributed to 20 people at a party. Five hats are red, five hats are blue, and 10 hats are purple. In how many different ways can this be done? (Assume the people are mingling and moving about.)

b) If the 20 people are clones and cannot be distinguished, in how many essentially different ways can these hats be distributed?

Question 10: A committee of five must be formed from five men and seven women.

a) How many committees can be formed if gender is irrelevant?

b) How many committees can be formed if there must be at exactly two women on the committee?

c) How many committees can be formed if one particular man must be on the committee and one particular woman must not be on the committee?

d) How many committees can be formed if one particular couple (one man and one woman) can't be on the committee together?

Question 11:

a) From 10 people k are needed for a committee. Write down a formula for the number of ways this can be done.

b) Suppose we want our formula to hold NO MATTER WHAT. Set $k = 11$ into your formula. What value should $(-1)!$ have so that your formula is correct for the number of ways to select 11 people from 10 for a committee?

Question 12: a) Suppose *a* and *b* are positive integers with $a+b=n$. Show that:

$$\binom{n}{a\ \ b} = \binom{n-1}{a-1\ \ b} + \binom{n-1}{a\ \ b-1}$$

b) Suppose *a*, *b* and *c* are three positive integers with $a+b+c=n$. Show that:

$$\binom{n}{a\ \ b\ \ c} = \binom{n-1}{a-1\ \ b\ \ c} + \binom{n-1}{a\ \ b-1\ \ c} + \binom{n-1}{a\ \ b\ \ c-1}$$

Question 13:

a) Twelve white dots lie in a row. Two are to be coloured red. In how many ways can this be done?
b) Consider the equation $10 = x+y+z$. How many solutions does it have if each variable is to be a positive integer or zero?

Question 14:

a) In how many ways can the letters ABCDEFGH be arranged?
b) In how many ways can the letters ABCDEFGH be arranged with letter G appearing somewhere to the left of letter D?
c) In how many ways can the letters ABCDEFGH be arranged with the letters F and H not adjacent?

Question 15: Let's establish the formulas from the textbooks …

a) Suppose *r* objects are to be selected from a collection of *n* objects with the order in which they are selected considered important. Use the labeling principle to show that this can be done in ${}_nP_r = \frac{n!}{(n-r)!}$ different ways.
b) Suppose *r* objects are to be selected from a collection of *n* objects without regard to order. Use the labeling principle to show that this can be done ${}_nC_r = \frac{n!}{r!(n-r)!}$ different ways.

NOW FORGET THESE FORMULAS. You never need them!

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

Question 16: a) In poker "three of a kind" is a set of three cards of the same value with neither of the two remaining cards that value (or of value equal to each other). What is the probability of being dealt three-of-a-kind?

b) What is the probability of being dealt "four of a kind" in poker?

Question 17:
a) Prove that the product of any 3 consecutive integers is sure to be divisible by 3! = 6.
b) Prove that the product of any 7 consecutive integers is sure to be divisible by 7! = 5040.
b) Prove that the product of any *k* consecutive integers is sure to be divisible by k!

<u>HINT</u>: Consider the problem: *k people from N are to be selected for a committee. In how many ways can this be done?* We know that the answer to this must be a whole number. Thus, $\frac{N!}{k!(N-k)!}$ is always an integer.

Question 18: Consider the question …

In how many different ways can 8 people sit around a round table?

This is a vague question. What does "different" mean?

a) Answer the question if the chairs of the table are marked North, Northeast, East, Southeast, South, …, Northwest.
b) Answer the question if the chairs are not marked so that two different rotations of the same arrangement of people would be considered the same.
c) Answer the question under the assumption that rotations are considered the same and reflections about a diameter of the table are considered the same.

Suppose two particular people must *not* sit next to one another. Answer each of the questions a), b) and c) with this added restriction.
(HINT: First count the number of arrangements with that couple seated together.)

CURRICULUM CONNECTIONS
Counting techniques; Permutations; Combinations

Question 19 (HARD): A "factorian" is a number that equals the sum of its digits factorialised. For example, 145 is a factorian since 1! + 4! + 5! = 145. The number 1 is a factorian, as is the number 2. (We have 1! = 1 and 2! = 2.) There is only one other factorian. What is it?

CHALLENGE: Prove that there are only four factorians.

Question 20:
An ice-cream stand offers the "mega-bowl special:" twelve-scoops in a bowl from a choice of twelve possible flavors. How many different mega-bowl combinations does it offer?

COMMENT: The problem here is that scoops, like the clones of question 9b), are indistinguishable. Moreover, you are not told how many scoops there are to be of a particular label (flavor). Problems like these are hard and fall under the category of what is called "multi-choosing."

HINT: Question 13 answers the following question; *Ten scoops of ice-cream sit in a bowl with each scoop one of three possible flavours. How many possibilities can occur?* Do you see how?

Question 21: (TRICKY)

a) In how many different ways can nine people be split into three groups of three?
b) In how many different ways can 20 people be split into two groups of six and two groups of four?
c) Prove that if k and N are positive integers, then so is $\frac{(kN)!}{k!(N!)^k}$.

Question 22: OPTIONAL CHALLENGE

a) Show that "factorials grow larger than the powers of ten." That is, show that $\frac{10^n}{n!} \to 0$ as $n \to \infty$.

<u>HINT</u>: To do this, first show that $\frac{10^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{10^n}{n!}$ for $n > 20$. Let $M = \frac{10^{20}}{20!}$.

Deduce that:

$$\frac{10^{21}}{21!} < \frac{M}{2}$$
$$\frac{10^{22}}{22!} < \frac{M}{4}$$
$$\frac{10^{23}}{23!} < \frac{M}{8}$$
$$\vdots$$

Now explain why $\frac{10^n}{n!} \to 0$.

b) Show that "factorials grow faster than tenth powers." That is, show that $\frac{n^{10}}{n!} \to 0$ as $n \to \infty$.

Question 23: In advanced mathematics the word *permutation* is used to mean a *rearrangement* or *reordering*. For example, a permutation of the list 12345 would be 42315.

a) How many different permutations are there of the list 12345?
b) How many of those permutations keep the number 3 fixed as the third number in place?
c) How many permutations of 12345 leave all but two numbers fixed in place?

When applied to a string of numbers, such as 12345, a permutation is called *even* if a large number appears to the left of a smaller number an even number of times, and odd otherwise. For example, for the permutation

23514

the number 2 appears to the left of 1, 3 appears to the left of 1, 5 appears to the left of 1 and 4. This permutation is even.

The permutation 34215 is odd, 54321 is even, and 12345 is also even.

d) How many even permutations are there of 12345?
e) How many odd permutations are there of 12345?
f) How many even permutations of 12345 leave 1 fixed in first place?
g) The numbers 1 through 100 are written in reverse order. Does this represent an even or odd permutation?

The action of swapping two elements in a list is called a *transposition*. For example, 13245 is a transposition of 12345, and 35124 is a transposition of 34125.

h) Explain the following: A transposition that swaps two adjacent entries in a list turns an even permutation into an odd permutation, and vice versa.

Let's call a transposition *elementary* if it swaps two adjacent entries. Any permutation of 12345 can be "returned" to this state via a series of elementary transpositions. For example:

$$35142 \to 31542 \to 13542 \to 13524$$
$$\to 13254 \to 12354 \to 12345$$

i) Convert the permutation 54321 into 12345 via a series of elementary transpositions. Count the number of transpositions you use to accomplish this feat.

j) Convert the transposition 52341 of 12345 into 12345 via a series of elementary transpositions. How many elementary transpositions did you use?

k) Convert the transposition 15342 of 12345 into 12345 via a series of elementary transpositions. How many elementary transpositions did you use?

l) Prove that any transposition of 12345 is the result of an odd number of elementary transpositions.

HINT: Consider a permutation …a…b…. If it takes *n* elementary transpositions to move a into b's place, show that it then takes *n* -1 elementary transpositions to move b into a's place.

m) Explain the following: Any transposition applied to an even permutation of 12345 turns it into an odd permutation, and vice versa.

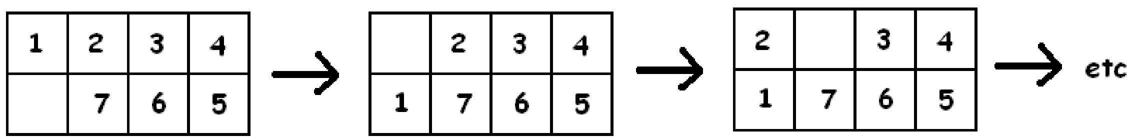
HINT: Use parts h) and l).

n) Explain the following: An even permutation of 12345 can be returned to this beginning state only via an even number of transpositions. An odd permutation of 12345 can be returned to this state only via an odd number of transpositions.

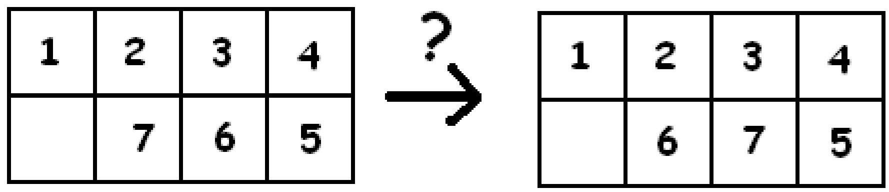
COMMENT: Some people take this final result as the definition of what it means for a permutation to be even or odd.

Question 24: OPTIONAL

a) Seven tiles, numbered 1 through 7. are held in a 2×4 frame. One cell is left blank. Tiles may slide horizontally and vertically into the blank cell to form rearrangements of the seven tiles.

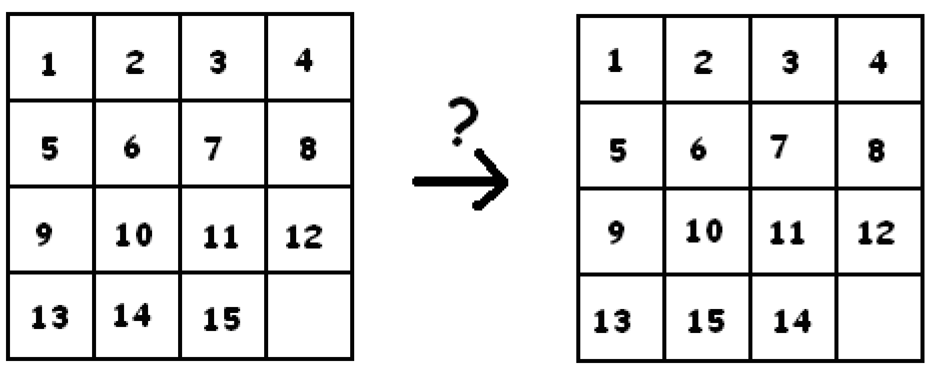


Construct a model of this apparatus (perhaps use playing cards). Is it possible to create an arrangement with tiles 6 and 7 swapped in place?



Explain/prove any claims you make.

b) In 1878 American puzzlist Sam Loyd introduced his *Boss puzzle*. It consisted of 15 tiles, numbered 1 thought 15, held in a 4×4 frame with one cell left blank. By sliding tiles can you solve the puzzle indicated below?



Explain your reasoning.