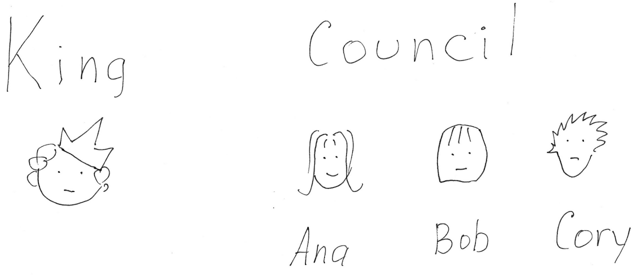
Voting Paradoxes

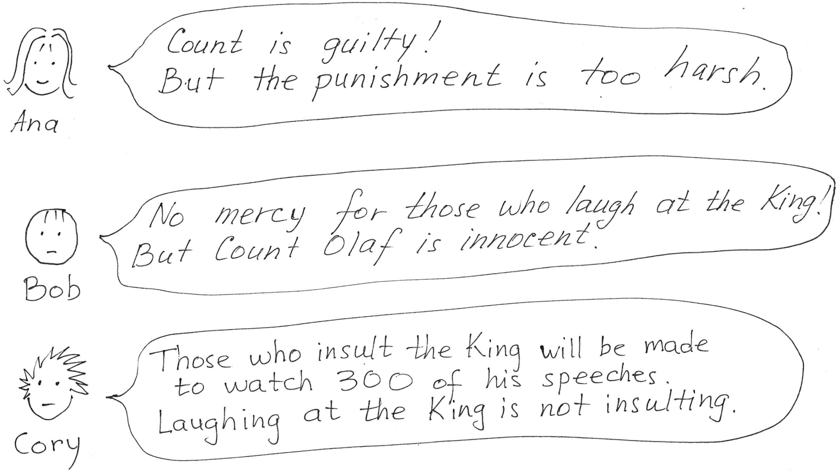
Elena Galaktionova

Mobile Math Circle

Part I

Constructing voting paradoxes with logic


King
Council
Ana
Bob
Cory


Ana
Count is guilty!
But the punishment is too harsh.
Bob
No mercy for those who laugh at the King!
But Count Olaf is innocent.
Cory
Those who insult the King will be made
to watch 300 of his speeches.
Laughing at the King is not insulting.

	Ana	Bob	Cory	majority
Laughing at the King is an insult	Y	Y	N	Y
Insult will be punished by making to watch 300 speeches	N	Y	Y	Y
Olaf laughed at the King	Y	N	Y	Y
Agree with all the above statements	N	N	N	N\Y

Rules for Voters:

1. You shall always vote: no abstaining.

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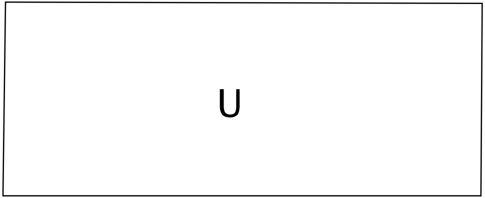
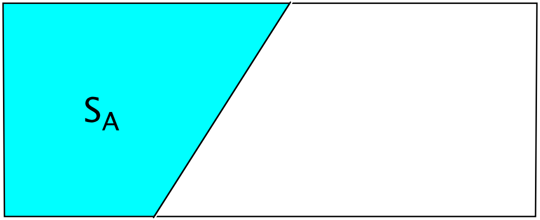
Rules for Voters:

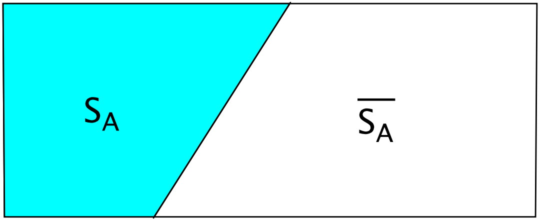
1. You shall always vote: no abstaining.
2. You shall not change your mind.
3. You shall be logically consistent.



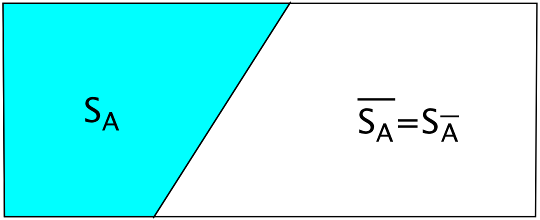
U represents the set of all voters.



S_A represents all voters that vote *for* proposition A.

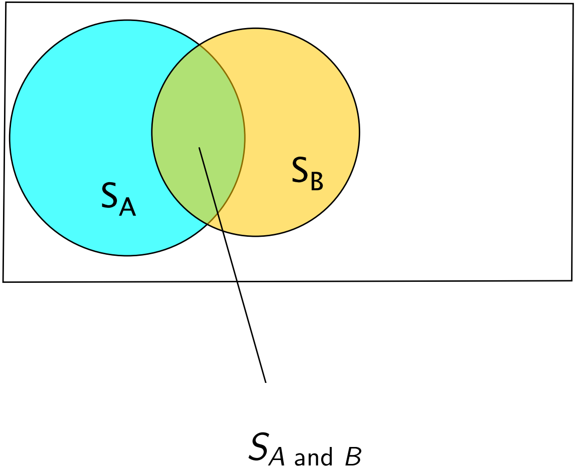


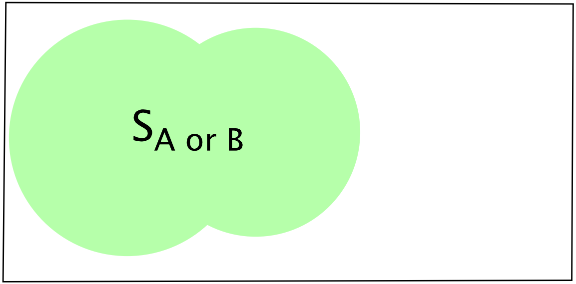
S_A represents all voters that vote *for* proposition A.
$\overline{S}$ denotes the complement of a set S.



S_A represents all voters that vote *for* proposition A.
$\overline{S}$ denotes the complement of a set S.
$\overline{A}$ is "not A", the negation of proposition A.

Logical consistency rules for voters:
1. If you vote *for* A then vote *against* $\overline{A}$. And vice versa.


S_A
S_B
$S_{A \text{ and } B}$



$S_{A \text{ or } B}$

Logical consistency rules for voters:

1. If you vote *for* A then you vote *against* $\overline{A}$;
2. you vote for "A and B" exactly when you vote for A *and* you vote for B;
3. you vote for "A or B" exactly when you vote for A *or* for B.

When can get we a contradiction?

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When
S_A is a majority and
S_B is a majority but
$S_{A \text{ and } B}$ is not.

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When
S_A is a majority and
S_B is a majority but
$S_{A \text{ and } B}$ is not.
This is possible exactly when there is no majority which votes unanimously on all propositions.

Theorem (Shapiro, 1995)

Suppose a council does not have a majority block which votes unanimously on all propositions. Then it is possible to have the council approve any given proposition.

Historic Interlude

Historic Interlude

Condorcet

Historic Interlude

Condorcet

Borda

Historic Interlude

Condorcet

Borda

Llull

Condorcet

Borda

Llull

Arrow

Part II

Constructing voting paradoxes with symmetry

Due to budget constraints, from now on, only one type of cookies will be served at the Mobile Math Circle meetings. However, the students are allowed to vote on the type of cookies to be served. Four choices were suggested:

a: Butter cookies b: Chips Ahoy!

c: Macadamia Nut cookies d: Raisin cookies

Voting Profile

ranking\ number of students	5	3	5	4
1 st preference	a	a	b	c
2nd	d	d	c	d
3rd	c	b	d	b
4th	b	c	a	a

Voting Profile

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1 st preference	a	a	b	c
2nd	d	d	c	d
3rd	c	b	d	b
4th	b	c	a	a

In **plurality method** the candidate with the most first-place votes wins.

Instant-runoff voting: Initially, only top choices are counted. The candidate in last place, i.e. with the least number of top votes, is eliminated from the race. The same method is repeated with the remaining candidates until a single candidate remains.

Pairwise comparison or the Condorcet criterion: If a candidate is preferred by the voters over each of the other candidates in a head-to-head comparison, then that candidateshould be the winner of the election, called **Condorcet winner**.

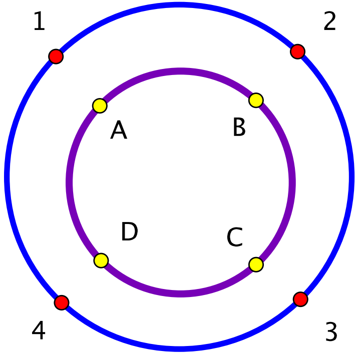
Voting Profile

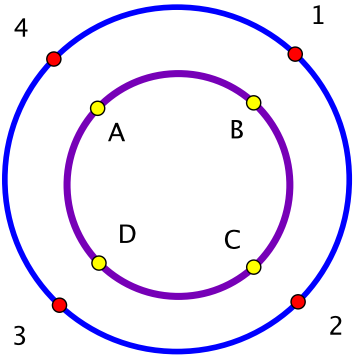
ranking\ number of students	5	3	5	4
1 st preference	a	a	b	c
2nd	d	d	c	d
3rd	c	b	d	b
4th	b	c	a	a

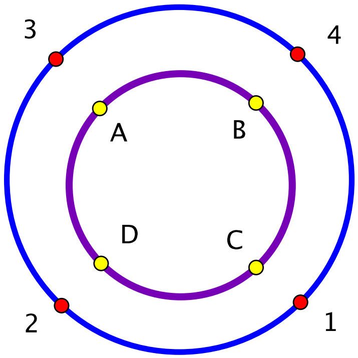
The Borda Count Method: A candidate is given 3 points for each first place on the list of individual preferences, 2 points for the second place, 1 point for the 3rd place and 0 points for the last place. The candidate with the highest total sum of points is the winner.

Ann, Bob, Cory and Don are the candidates for a position in your class. 21 students will vote. Here are their preferences for the candidates:

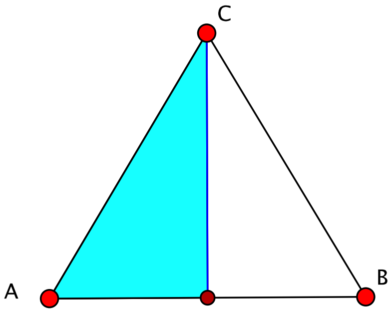
10 students: $A \succ B \succ C \succ D$
6 students: $B \succ C \succ D \succ A$
5 students: $C \succ D \succ A \succ B$


1
2
A
B
D
C
4
3


4
1
A
B
D
C
3
2

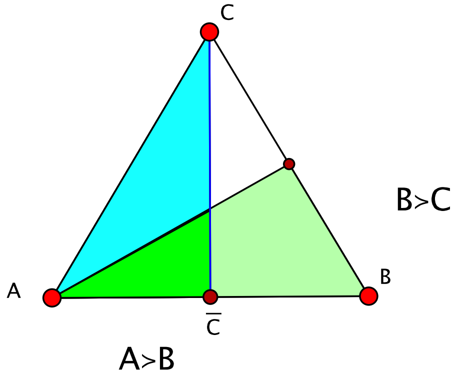

3
4
A
B
D
C
2
1

Three candidate vote geometric representation (Saari)

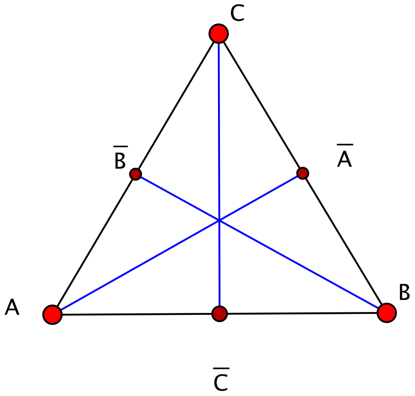


A≻B

Geometric representation



Geometric representation



4 voters: $A \succ B \succ C$

5 voters: $B \succ C \succ A$

1 voter: $C \succ A \succ B$

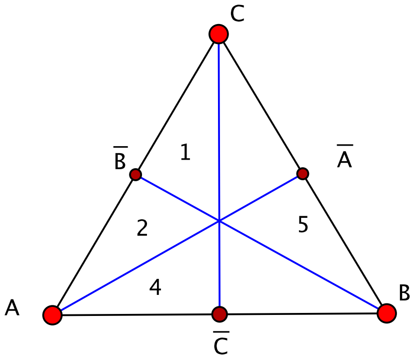
2 voters: $A \succ C \succ B$

4 voters: $A \succ B \succ C$

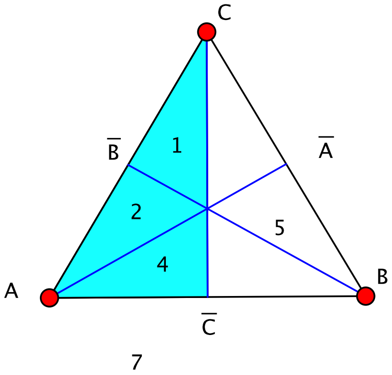
5 voters: $B \succ C \succ A$

1 voter: $C \succ A \succ B$

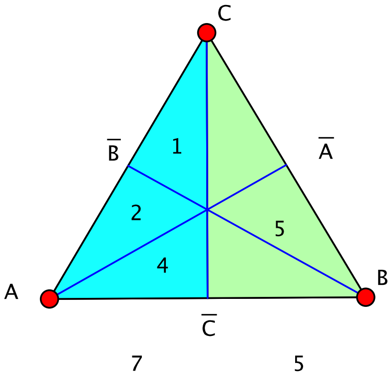
2 voters: $A \succ C \succ B$



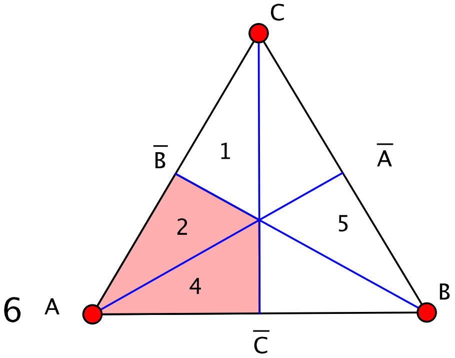
Geometric representation



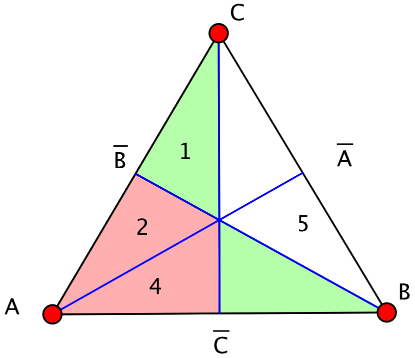
Geometric representation



Geometric representation



Geometric representation



6

13

6 voters: $A \succ B \succ C$

4 voters: $B \succ A \succ C$

6 voters: $B \succ C \succ A$

2 voters: $C \succ B \succ A$

6 voter: $C \succ A \succ B$

3 voters: $A \succ C \succ B$

6 voters: $A \succ B \succ C$

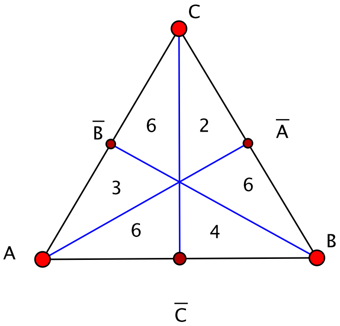
4 voters: $B \succ A \succ C$

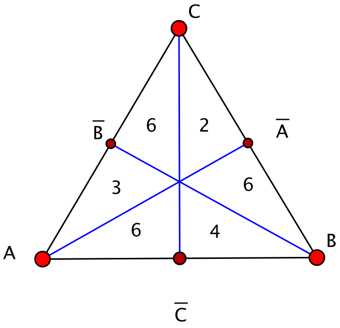
6 voters: $B \succ C \succ A$

2 voters: $C \succ B \succ A$

6 voter: $C \succ A \succ B$

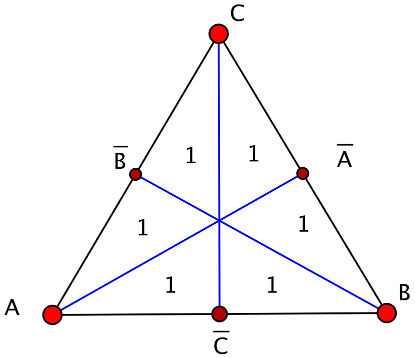
3 voters: $A \succ C \succ B$





After a successful campaign by candidate A three voters changed their preferences from $B \succ A \succ C$ to $A \succ B \succ C$ and two voters changed their ranking from $C \succ B \succ A$ to $C \succ A \succ B$. Draw the new voting profile.

Kernel



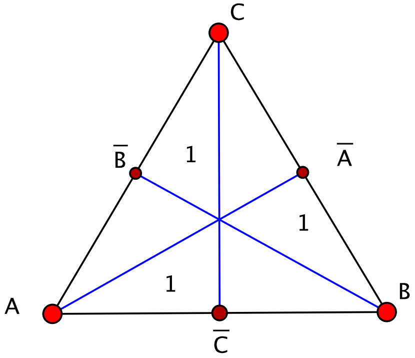
Symmetric Voting Profiles

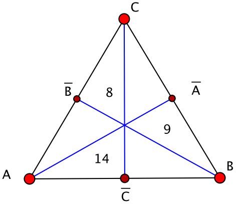
Condorcet

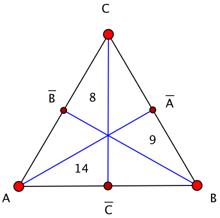
$A \succ B \succ C$

$B \succ C \succ A$

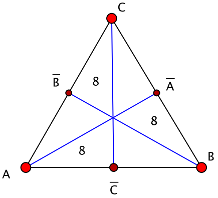
$C \succ A \succ B$



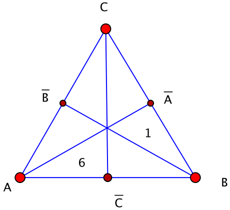

C
B̄
8
Ā
9
14
A
B
C̄

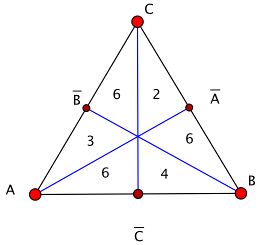

C
$\bar{B}$
8
$\bar{A}$
9
14
A
$\bar{C}$
B

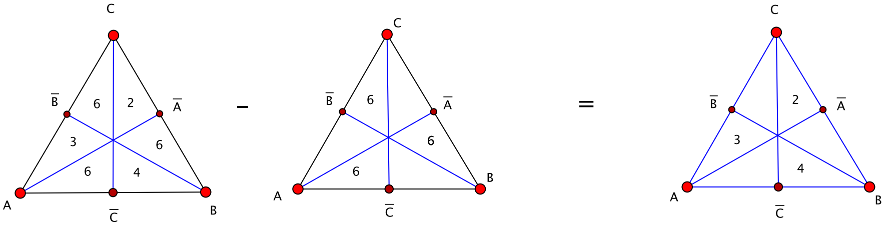
−


C
$\bar{B}$
8
$\bar{A}$
8
8
A
$\bar{C}$
B

=


C
$\bar{B}$
$\bar{A}$
1
6
A
$\bar{C}$
B


C
B̄
6
2
Ā
3
6
6
4
A
B
C̄


C
B̄
6
2
Ā
3
6
6
4
A
C̄
B
−
C
B̄
6
Ā
6
6
A
B
C̄
=
C
B̄
2
Ā
3
4
A
B
C̄

Symmetric Voting Profiles

Reversal

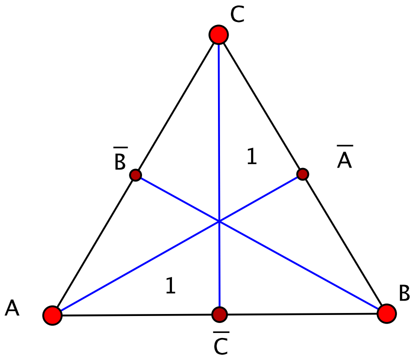
$A \succ B \succ C$

$C \succ B \succ A$

Reversal

$A \succ B \succ C$

$C \succ B \succ A$



Saari (1999): Condorcet component is responsible for many paradoxes!

Any discrepancies between the Borda Count ranking outcome and the pairwise outcome are due to a Condorcet cycle component.

Any discrepancies between the Borda Count outcome and the plurality outcome are due to Reversal components.

Part III

Arrow's Theorem

Unanimity If every voter prefers the candidate X to the candidate Y then X will rank above Y in the outcome.

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Transitivity If X ranks above Y and Y ranks above Z in the outcome then X ranks above Z in the outcome. (Short-hand: If $X \succ Y$ and $Y \succ Z$ then $X \succ Z$ in the outcome.) If X ties with Y and Y ties with Z in the outcome, then X ties with Z in the outcome. (Short-hand: If $X \sim Y$ and $Y \sim Z$ then $X \sim Z$.)

Keebler Chips Deluxe 32%, Chips Ahoj! 31%, Oatmeal Raisin 37%

Unanimity If every voter prefers the candidate X to the candidate Y then X will rank above Y in the outcome.

Transitivity If $X \succ Y$ and $Y \succ Z$ then $X \succ Z$ in the outcome. If $X \sim Y$ and $Y \sim Z$ then $X \sim Z$ in the outcome.

Independence of Irrelevant Alternatives (IIA): removal of a candidate should not affect the relative ranking of the other two candidates in the outcome of the vote. That is, the ranking of the candidates X and Y by the voting system depends only on the ranking of X and Y by voters and does not depend on rankings of Z in the voting profile.

Winning Set

Suppose that for a particular profile the voting system outcome is $A \succ B$. According to IIA this depends only on $A : B$ ranking of voters. We say that the set M of all voters which ranked A above B *wins for* A *over* B(for $A \succ B$).

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Suppose M wins for $A \succ B$. Consider a profile: voters in M all vote $A \succ B \succ C$
voters not in M vote $B \succ C \succ A$.

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voters in M all vote $A \succ B \succ C$
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If M wins for $A \succ B$ it wins for $A \succ C$.

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Reverse the ratings in the above profile to obtain:
if a set N wins for $B \succ A$ then it wins for $C \succ A$.

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Reverse the ratings in the above profile to obtain:
if a set N wins for $B \succ A$ then it wins for $C \succ A$.
We can talk about a winning set.

No ties.

Assume that for some voting profile the outcome is $A \sim B$.

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Consider a profile where
voters in M all vote $A \succ B \succ C$
voters not in M vote $B \succ C \succ A$.

Corollary

For any set of voters M either M or it's complement $\overline{M}$ is a winning set.

Let M and N be two winning sets. Consider a profile where voters in M rank $A \succ B$, voters in $\overline{M}$ rank $B \succ A$;
voters in N rank $B \succ C$, voters in $\overline{N}$ vote $C \succ B$.

Intersection of winning sets is a winning set

Let M and N be two winning sets. Consider a profile where voters in M rank $A \succ B$, voters in $\overline{M}$ rank $B \succ A$;
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By transitivity $A \succ C$.

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Construct such profile with an additional condition that only voters in $M \cap N$ rank $A \succ C$.

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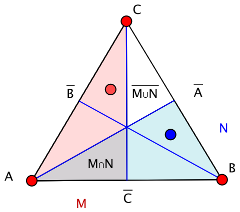

Corollary

There are no disjoint winning sets.

There exists a *dictator*, i.e. a distinguished voter v so that $\{v\}$ is a winning set and all other winning sets are exactly the sets which contain v.

If a voting system with three or more candidates satisfies unanimity, transitivity and IIA then it is a dictatorship.

Math Cheat

A voting profile for the ranking of three candidates is a point in a six dimensional space. D. Saari introduced the following four pairwise orthogonal subspaces which span this six-dimensional space:

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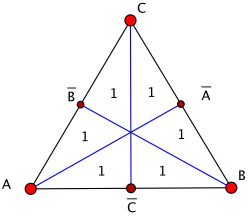
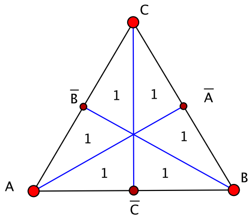
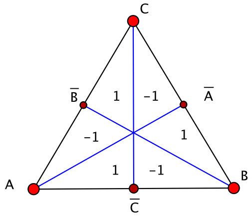


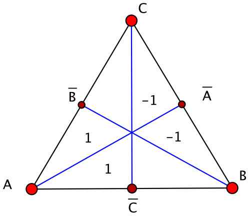
one dimensional Kernel subspace spanned by



one dimensional Condorcet subspace spanned by

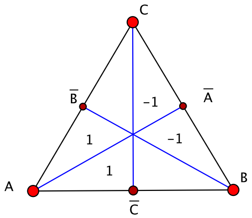


two dimensional Basic subspace spanned by



A-Basic B_A

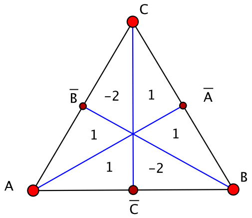
two dimensional Basic subspace spanned by



A-Basic B_A

also B_B and B_C.

two dimensional Reversal subspace spanned by



A-Reversal R_A

also R_B and R_C.

Theorem (Saari, 1999)

1. On the Basic subspace plurality, Borda Count and pairwise rankings agree.

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2. Adding a non-zero element of the Condorcet subspace does not change plurality or Borda Count outcomes, but adds a cycle to the pairwise ranking.

Theorem (Saari, 1999)

1. On the Basic subspace plurality, Borda Count and pairwise rankings agree.
2. Adding a non-zero element of the Condorcet subspace does not change plurality or Borda Count outcomes, but adds a cycle to the pairwise ranking.
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4. (Arrow's Possibility Theorem) Consider the set of all voting profiles with no Condorcet component (i.e. the five dimensional subspace orthogonal to Condorcet subspace). Voting systems on this set which satisfy transitivity, unanimity and IIA include Borda Count, pairwise ranking and some other methods.

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