Constructing voting paradoxes with logic and symmetry

Part I: Voting and Logic

Problem 1. There was a kingdom once ruled by a king and a council of three members: Ana, Bob and Cory. It was a very democratic monarchy. The council decided on all important issues by voting. The King could not vote, but he was the only one who could bring propositions to vote. For example, he could call the council meeting in the morning and have them vote on whether the King should have a nap after lunch.

One day, as the King was coming up onstage to deliver his speech on the National Cherry Pie Day, he tripped and his crown fell off. In the awkward silence that followed, the King heard Count Olaf's laughter. His Majesty got very upset. He has to punish the Count for this insulting laughter! He thought of a punishment and decided on making Count Olaf watch the videos of the King's last three hundred speeches.

This important decision needed approval of the council. King privately talked to each member of the council only to find out that no one supports him. Ana was appalled by Count's behavior. However, when she heard of punishment, she paled and begged the King for mercy. "This is too cruel!" - said Ana. "Should Count Olaf be guilty of such behavior, he would have to be prosecuted with all strictness. However, I heard Olaf. He was just coughing. He is innocent!" - said Bob. Cory was of dangerous opinion that laughing at the King is not an insult, although he agreed that having to watch three hundred of King's speeches is an appropriate punishment for those who insult the King. In short, everyone was against the King's proposition. However, when the King brought the issue before the council to vote, Olaf's punishment was approved, with all members voting according to their opinions. What did the King do?

Problem 2.* This problem fixes a small gap in our argument in support of Shapiro's Theorem. There is a council of three voters: Xavier (X), Yelena (Y) and Zane (Z). You know that the votes of X and Y agree on all propositions, except proposition A and, necessarily, some of it's logical derivatives, such as $\overline{A}$, "A and B" etc. X is against A while Y and Z are for it. Construct propositions to vote on which result in the approval of $\overline{A}$.

Part II: Voting paradoxes and symmetry

Problem 3. Due to budget constraints, from now on, only one type of cookies will be served at the Mobile Math Circle meetings. However, the students are allowed to vote on the type of cookies to be served. Four choices were suggested:

a: Butter cookies b: Chips Ahoy! c: Macadamia Nut cookies d: Raisin cookies

A poll was taken where each student ranked the four types of cookies in order of preference. Here is the result of the poll:

ranking\ number of students	5	3	5	4
1 st preference	a	a	b	c
2nd	d	d	c	d
3rd	c	b	d	b
4th	b	c	a	a

These data is called a **voting profile**. How should we select a winner? Here are some popular methods:

In **plurality method** the candidate with the most first-place votes wins.

Instant-runoff voting: Initially, only the top choices are counted. The candidate in last place, i.e. with the least number of top votes, is eliminated from the race. Then the candidates ranked behind the eliminated candidate move up one place. The same method is repeated again until a single candidate remains.

Pairwise comparison or the Condorcet criterion: If a candidate is preferred by the voters over each of the other candidates in a head-to-head comparison, then that candidate should be the winner of the election, called the **Condorcet winner**.

The Borda Count Method: A candidate is given 3 points for each first place on the individual rankings, 2 points for the second place, 1 point for the 3rd place and 0 points for the last place. The candidate with the highest total sum of points is the winner.

Notation for ranking: A$\succ$ B means A ranks higher than B. A tie is denoted $A \sim B$.

Task: Find the winner as well as the complete ranking for each method.

(Note: This problem is made up! Not to worry, we'll keep serving variety of cookies.)

Problem 4. Let us introduce the **tournament method:** Put candidates names in some order. Vote between the first two candidates. The loser is eliminated and the winner goes on to compare with the third candidate, etc.

Consider the following scenario: Ann, Bob, Cory and Don are the candidates for a position in your class. 21 students will vote. Here are their preferences for the candidates:

10 students: $A \succ B \succ C \succ D$

6 students: $B \succ C \succ D \succ A$

5 students: $C \succ D \succ A \succ B$

a) Organize the tournament method vote in such a way that Don wins.

b) A voting method satisfies the property of **unanimity** if whenever every voter ranks candidate X higher than candidate Y, the outcome of the vote should rank X higher than Y. Does the Tournament Method satisfy unanimity?

Three candidate case.

Problem 5. In how many ways can one rank three candidates A, B, C (no ties allowed)?

Problem 6. *Geometric representation of a voting profile.*

Let us represent the three candidates A, B, C as the vertices of an equilateral triangle. An interior point P of this triangle represents a voter's preference based on the distance from P to the vertices: the one closer to P is ranked higher. If the distance to both vertices is the same then P represents a tie between the two candidates.

a) Find the locus of points representing $A \sim B$.

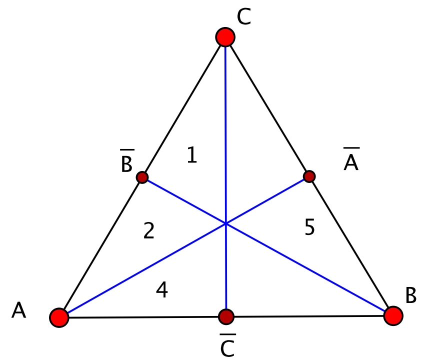
b) Find the set of all points within the triangle representing $A \succ B$.

c) Find the set of all points within the triangle representing $A \succ B \succ C$.

(Notice that if a vote is in the area $A \succ B$ and in the area $B \succ C$ then it is also in the area $A \succ C$.)

Example. Draw geometric representation using equilateral triangle for the voting profile:

4 voters: $A \succ B \succ C$
5 voters: $B \succ C \succ A$
1 voter: $C \succ A \succ B$
2 voters: $A \succ C \succ B$



Problem 7. For the voting profile above write
a) pairwise tallies: the total number of $A \succ B$ votes under the segment $A\overline{C}$, etc. ;
b) the number of top choice votes by each vertex;
c) the Borda Count below this number.

Problem 8. *Paradox: failure of positive association.*
a) Use equilateral triangle to represent the following profile:
6 voters: $A \succ B \succ C$
4 voters: $B \succ A \succ C$
6 voters: $B \succ C \succ A$
2 voters: $C \succ B \succ A$
6 voter: $C \succ A \succ B$
3 voters: $A \succ C \succ B$

b) Using instant run-off method for this profile, who wins?

c) After a successful campaign by candidate A three voters changed their preferences from $B \succ A \succ C$ to $A \succ B \succ C$ and two voters changes their ranking from $C \succ B \succ A$ to $C \succ A \succ B$. Draw the new voting profile.

d) Who is the instant run-off winner now?

Problem 9. a) (non-transitivity, or Condorcet cycle) Construct a voting profile with 14 top choice votes for A, 8 top choice votes for B, 9 top choice for C and with pairwise ranking $A \succ B$, $B \succ C$ and $C \succ A$.

Hint: use representation triangle.

b) (Reversal) Construct a voting profile with 14 top choice votes for A, 8 top choice votes for B, 9 for C (so that plurality ranking is $A \succ B \succ C$), but with Borda Count ranking $C \succ B \succ A$.

Problem 10. *Contribution of symmetries.*

a) What are the profiles whose triangle representation has the most symmetries?

What high symmetry profile means for candidates comparison? What should be the outcome of the election for such profile?

What are the results of plurality, Borda Count and Pairwise votes on such profile?

Examine how voting outcomes are affected by adding such profile to a given profile.

b) Find another profile where there is a symmetry between all three candidates.

c) Find the results of plurality, pairwise comparison and Borda count for a Condorcet profile.

d) How does addition of a Condorcet profile to an existing profile affects the results of plurality? Borda Count? Condorcet?

e) Find the results of plurality, pairwise comparison and Borda count for a Reversal profile.

f) How does addition of this profile to an existing profile affects the results of plurality? Borda Count? Condorcet?

Problem 11. Subtract the largest Condorcet profile from the profile you constructed in Problem 9a). Use pairwise method to determine group ranking. Is there a cycle?

Problem 12. Subtract the largest Condorcet profile from the profile in Problem 8a). Solve the problem with the new profile. Is there a paradox?

Problem 13.* a) Suppose in a three-candidate situation pairwise comparison results in a Condorcet cycle. Show if we use a tournament method and start by comparing candidates X and Y and then between the winner and Z, then Z always wins.
b) Show that if there is no Condorcet cycle then there is a Condorcet (pairwise comparison) winner who will always win with tournament procedure.

Part III : Arrow's Theorem

Problem 14. Suppose that in the scenario of Problem 3 it turned out that there is a lot of chocolate chips cookies lovers and also a significant number of oatmeal raisin cookie lovers. As a result, three candidates emerge with the top choice votes split

Keebler Chips Deluxe 32%, Chips Ahoy! 31%, Oatmeal Raisin 37%.

a) Which type of cookies is the plurality winner?

b) Suppose that Keebler Chips Deluxe are discontinued. Then the vote is between the two remaining types of cookies. Which type is the winner now?

Independence of Irrelevant Alternatives (IIA): removal of a candidate should not affect the relative ranking of the other two candidates in the outcome of the vote. That is, the ranking of the candidates X and Y by the voting system depends only on the ranking of X and Y by voters and does not depend on rankings of Z in the voting profile.

c) Does the plurality method satisfy the Independence of Irrelevant Alternatives property?

Voting axioms:

Unanimity If every voter prefers the candidate X to the candidate Y then X will rank above Y in the outcome.

Transitivity If X ranks above Y and Y ranks above Z in the outcome then X ranks above Z in the outcome. (Using short-hand notation: If $X \succ Y$ and $Y \succ Z$ then $X \succ Z$ in the outcome.) If X ties with Y and Y ties with Z in the outcome, then X tiex with Z in the outcome. (Short-hand: If $X \sim Y$ and $Y \sim Z$ then $X \sim Z$.)

Independence of Irrelevant Alternatives If a choice is removed, then the relative ranking of remaining candidates should not change.

In the remaining problems we consider only three-candidate voting systems which satisfy transitivity, unanimity and IIA.

Winning set. Suppose that for a particular profile the voting system outcome is $A \succ B$. According to IIA this depends only on $A : B$ ranking of voters. We say that the set M of all voters which ranked A above B in this profile *wins for A over B*(for $A \succ B$). By IIA whenever all voters in M rank $A \succ B$ and all voters outside M rank $B \succ A$, then $A \succ B$ in the outcome.

Problem 15. a) Does the set of all voters wins for a candidate over another? Why?

b) Can it be that an empty set wins for A over B?

Problem 16. *Set that wins for one wins for all.*

a) Show that if a set M wins for $A \succ B$ then it wins for $A \succ C$.

b) Similarly, show that if a set N wins for $B \succ A$ then it wins for $C \succ A$. Hint: reverse the ratings in your proof of part a).

c) Show that if a set wins for $A \succ B$ then it wins for all 6 possible ratings of pairs.

Thus, it makes sense to talk about a winning set.

Problem 17. *No ties.*

Show that there cannot be ties in the outcome of voting.

Hint: assume that for some voting profile the outcome is $A \sim B$. Consider the set M of all voters in this profile who rank $A \succ B$. Construct a voting profile which will lead to contradiction with the axioms.

Problem 18. Corollary. For any set of voters M either M or it's complement $\overline{M}$ is a winning set.

Problem 19. *Intersection of winning sets is a winning set.*

Let M and N be two winning sets. Consider a profile where voters in M rank $A \succ B$, voters in $\overline{M}$ rank $B \succ A$; voters in N rank $B \succ C$, voters in $\overline{N}$ vote $C \succ B$.

a) What can you say about the $A : C$ ranking in the outcome?

b) Construct such profile with an additional condition that only voters in $M \cap N$ rank $A \succ C$. Hint: use representation triangle.

Problem 20. Corollary. There are no disjoint winning sets.

Problem 21. There exists a *dictator*, i.e. a distinguished voter v so that $\{v\}$ is a winning set and all other winning sets are exactly the sets which contain v.

Problem 22. The voting system in the previous problem is the same as following: there is a voter v such that the voting outcome coincides with the ranking given by v ignoring the input of other voters. Such system is called *a dictatorship.*

Problem 23.* In the original statement of the theorem Arrow allows ties in the voting profile. Explain why the conclusion of the theorem still holds true for this case.

References:

Part I *Voting and Logic* : Problem 1 is a version of so-called "Doctrinal Paradox" or "Discursive Dilemma". Shapiro's theorem is in A. Shapiro *Logic and Parliament* (1995), Kvant, 1995, 3 (in Russian).

Part II *Voting and Symmetry* : Geometric representation of a 3 candidate voting profile, as well as contribution of symmetries are explained in
D.G. Saari, *Explaining all three-alternative voting outcomes*, Journal of Economic Theory 87, 313 - 355 (1999).

Part III *Arrow's Theorem*: The proof is an amalgam of the proofs of the Arrow's Theorem by Sridhar Ramesh (https://pleasantfeeling.wordpress.com/2009/04/19/arrowstheorem/) and by Terrence Tao (https://www.math.ucla.edu/ tao/arrow.pdf).