Inequalities

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1. QM-AM-GM-HM Inequalities

Many inequalities follow from the obvious

$$x_1^2 + x_2^2 + ... + x_n^2 \geq 0$$

For example: $a^2 + b^2 \geq 2ab$ is equivalent to $(a-b)^2 \geq 0$: true for any real numbers a, b. Similarly we prove that $a^2+b^2+c^2 \geq ab+ac+bc$ (how?). If we consider nonnegative a and b then the basic AM-GM inequality, $\frac{a+b}{2} \geq \sqrt{ab}$, is equivalent to $(\sqrt{a}-\sqrt{b})^2 \geq 0$.

Problem 1. Prove that for any $a, b > 0$

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

Problem 2. If $a_i > 0$ for $i = 1, 2, ..., n$, and $a_1...a_n = 1$, prove that

$$(1+a_1)(1+a_2)...(1+a_n) \geq 2^n.$$

General form of the *QM-AM-GM-HM inequalities*:

$$\sqrt{\frac{a_1^2+...+a_n^2}{n}} \geq \frac{a_1+...+a_n}{n} \geq \sqrt[n]{a_1...a_n} \geq \frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}.$$

An equality holds if and only if $a_1 = ... = a_n$.

Problem 3. For $a, b, c, d > 0$, prove that

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[3]{\frac{abc+abd+acd+bed}{4}}$$

Problem 4. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

for any nonnegative a, b, c.

The QM-AM-GM-HM inequality is a particular case of the *Power Mean Inequality*. Namely, if for positive $a_1, ..., a_n$ we set

$$P_r(a_1, ..., a_n) := \left(\frac{a_1^r + ... + a_n^r}{n}\right)^{1/r}$$

for $r \neq 0$, and $P_0(a_1, ..., a_n) := \sqrt[n]{a_1 ... a_n}$, $P_\infty := \max\{a_1, ..., a_n\}$, $P_{-\infty} := \min\{a_1, ..., a_n\}$, we have:

$$P_r(a_1, ..., a_n) \leq P_s(a_1, ..., a_n) \text{ for } r < s.$$

2. Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality: For any real numbers $a_1, ..., a_n$ and $b_1, ..., b_n$

$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) \geq (a_1 b_1 + ... + a_n b_n)^2$$

(equivalently, $|\vec{a}||\vec{b}| \geq |\vec{a} \cdot \vec{b}|$). When does the equality hold?

Problem 5. Prove the triangle inequality

$$\sqrt{a_1^2 + ... + a_n^2} + \sqrt{b_1^2 + ... + b_n^2} \geq \sqrt{(a_1 + b_1)^2 + ...(a_n + b_n)^2}.$$

Problem 6. If $a, b, c > 0$ prove that

$$abc(a + b + c) \leq a^3 b + b^3 c + c^3 a.$$

Problem 7. (IMO 1995) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}.$$

3. Rearrangement and Chebyshev's Inequalities

Rearrangement Inequality: If $a_1 \geq ... \geq a_n$ and $b_1 \geq ... \geq b_n$ then for any permutation (rearrangement) $c_1, ..., c_n$ of $b_1, ..., b_n$,

$$a_1 b_n + ... + a_n b_1 \leq a_1 c_1 + ...a_n c_n \leq a_1 b_1 + ... + a_n b_n$$

Chebyshev's Inequality: If $a_1 \geq ... \geq a_n$ and $b_1 \geq ... \geq b_n$ then

$$\frac{a_1 + ... + a_n}{n} \frac{b_1 + ... + b_n}{n} \leq \frac{a_1 b_1 + ... + a_n b_n}{n}$$

Problem 7. (IMO 1975) We consider two sequences of real numbers $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$. Let $z_1, z_2, \ldots, z_n$ be a permutation of the numbers $y_1, y_2, \ldots, y_n$. Prove that

$$\sum_{i=1}^{n} (x_i - \ y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2$$

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Problem 8. (IMO 1978) Let $a_1, a_2, ...$ be a sequence of pairwise distinct positive integers. Then

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

for any n.

Problem 9. Let $a, b, c > 0$ Prove that

(a) $abc \geq (a + b - c)(b + c - a)(c + a - b)$.

(b) $a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$.

4. More problems on inequalities

Problem 10. (IMO 1969) Given real numbers $x_1, ..., x_n$, $y_1, ..., y_n$, and $z_1, ..., z_n$ satisfying $x_i > 0, x_i y_i > z_i^2$, for $i = 1, ..., n$, prove that:

$$\frac{n^3}{(x_1 + ... + x_n)(y_1 + ... + y_n) - (z_1 + ... + z_n)^2} \leq \frac{1}{x_1 y_1 - z_1^2} + ... + \frac{1}{x_n y_n - z_n^2}.$$

Give necessary and sufficient conditions for equality.

Problem 11. Let a, b, c be real numbers such that $0 \leq a, b, c \leq 1$. Prove that

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq 2.$$

Problem 12. Let a, b, c be sides of a triangle. Prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3.$$

Problem 13. (USAMO 2003) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8$$

Problem 14. (USAMO 2001) Let a, b, and c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2$$