Polynomials and Complex Numbers

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1 Warming up problems

In the following two problems one may use the fact that if z is a root of a polynomial $P(z)$ and $P(z) = P(1/z)$, then $1/z$ is also a root of $P(z)$.

Problem 1. Solve the equation $z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0$.

Problem 2. Solve the equation

$$4z^{11}+4z^{10}-21z^9-21z^8+17z^7+17z^6+17z^5+17z^4-21z^3-21z^2+4z+4 = 0.$$

Problem 3. (a) Find a polynomial with integer coefficients whose zeros include $\sqrt{2} + \sqrt{5}$.

(b) Find a polynomial with integer coefficients whose zeros include $\sqrt{2} + \sqrt[3]{5}$.

Remark. Obviously Problem 3 (b) is substantially more difficult than Problem 3 (a). What would be the minimal degree of a polynomial one of the roots of which is $\sqrt{2} + \sqrt[3]{5} + \sqrt[5]{7}$?

Problem 4. Determine a, b, so that $(x - 1)^2$ divides $ax^4 + bx^3 + 1$.

2 Division with quotient and remainder

Division of polynomials. For any polynomials $f(x)$ and $g(x)$ there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x), \quad \deg r < \deg g \ \text{ or } r(x) = 0.$$

For example, if $f(x) = x^7 - 1$ and $g(x) = x^3 + x + 1$ then the quotient $q(x)$ is $x^4 - x^2 - x + 1$ and the remainder $r(x)$ is $2x^2 - 2$. In the case $g(x) = x - a$ we obtain an important fact: $f(a) = 0$ if and only if $f(x) = (x - a)q(x)$ for some polynomial $q(x)$

The coefficients of the polynomials can be in $\mathbb{C}, \mathbb{R}, \mathbb{Q}$, or $\mathbb{Z}$. In the case of $\mathbb{Z}$ we may have a situation when the quotient and the remainder are not with interger coefficients. Take for example $f(x) = x^2$ and $g(x) = 2x + 1$. Is there any such problem with $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$?

Problem 5. Find the remainder of $x^{81} + x^{49} + x^{25} + x^9 + x$ when divided by $x^3 - x$.

Problem 6. Find the remainder of x^{1959} when divided by $(x^2+1)(x^2+x+1)$.

Problem 7. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder of $f(x^5)$ when divided by $f(x)$.

Problem 8. Let $p(x)$ be a polynomial with integer coefficients. Assume that $p(a) = p(b) = p(c) = -1$, where a, b, c are three different integers. Prove that $p(x)$ has no integer zeros.

Problem 9. Let $P(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ be a polynomial with integer coefficients. Suppose that there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

Problem 10. (USAMO 1975) If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k + 1)$ for $k = 0, 1, 2, ..., n$, determine $P(n + 1)$.

3 Polynomial equations

Problem 11. Find all polynomials $P(x)$ for which $xP(x-1) = (x+1)P(x)$.

Problem 12. Determine all polynomials $P(x)$ such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

Problem 13. Find all polynomials $P(x)$, for which $P(x)P(2x^2) = P(2x^3 + x)$.

Problem 14. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all complex numbers z. Prove that $f(z)+f(-z) = 0$ for all complex numbers z.

Problem15. Find all polynomials $P(x)$, for which $P(x)P(2x^2) = P(2x^3+x)$.

Problem 16. Find all polynomials $P(x)$, for which $P(x^2)+P(x)P(x+1) = 0$.

4 Irreducibility of polynomials

Problem 17. Factor the following polynomials as products of irreducible polynomials with integer coffeicients.

(a) $x^4 + x^2 + 1$, (b) $x^{10} + x^5 + 1$, (c) $x^9 + x^4 - x - 1$.

Problem 18. Prove that $(1 + x + ... + x^n)^2 - x^n$ is the product of two polynomials.

Problem 19. If $a_1, ..., a_n$ are distinct integers, prove that $(x-a_1)...(x-a_n)-1$ is irreducible.