Power of a Point

Dimitar Grantcharov, Mid-Cities Math Circle

1. Power of a Point

Question: What necessary and sufficient conditions do we know for four points A, B, C, D to be concyclic (i.e. to lie on a common circle)?

Problem 1. (Power of a Point Theorem) Let k be a fixed circle with center O and radius r, and P be fixed point in the plane. A line ℓ through P intersects k at A and B. Prove that the product $PA \cdot PB$ depends only on P and k, but not on the line ℓ. Express $PA \cdot PB$ in terms of P and $k(O, r)$.

Remark. The product $PA \cdot PB$ can be understood as a signed product. What does that mean?

Definition. If P is a point, and $k(O, r)$ is a circle with center O and radius r in the plane, then $OP^2 - r^2$ is called *the power of P with respect to k*.

Problem 2. If the lines AB and CD meet at P and satisfy the (signed) identity $PA \cdot PB = PC \cdot PD$, then A, B, C, D are concyclic.

Problem 3. (ARML) In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

Problem 4. Square $ABCD$ of side length 10 has a circle inscribed in it. Let M be the midpoint of AB. Find the length of that portion of the segment MC that lies outside of the circle.

Problem 5. Let BD be the angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that $AE = CF$.

Problem 6. (IMO 1995) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

Problem 7. (USAMO 1998) Let k_1 and k_2 be concentric circles, with k_2 in the interior of k_1. From a point A on k_1 one draws the tangent AB to k_2 ($B \in k_2$). Let C be the second point of intersection of AB and k_1, and let D be the midpoint of AB. A line passing through A intersects k_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio $AM : MC$.

2. Radical Axes

Question: Given two circles, one with center O_1 and radius r_1, the other with center O_2 and radius r_2, what is the set of points (locus) with equal power with respect to the two circles? Describe this set for all cases of the circles (intersecting, tangent, nonitersecting).
Answer: The *radical axis* of the two circles.

Problem 8. Let k_1, k_2, k_3 be three circles in the plane. Prove that the radical axes of k_1 and k_2, of k_2 and k_3, and k_1 and k_3, either all coincide, or are concurrent (or parallel).

Problem 9. Suppose that $ABCD$ and $CDEF$ are cyclic quadrilaterals, and that the lines AB, CD, EF are concurrent. Then $EFAB$ is also cyclic. If A, D, and E are collinear, hen there is one more cyclic quadrilateral - which one is it?

Problem 10. (IMO 1997) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.

Problem 11. (IMO 1985) A circle with center O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangle ABC and KBN intersect at exactly two distinct points B and M. Prove that angle $\angle OMB$ is a right angle.

Problem 12. A quadrilateral $ABCD$ is inscribed in a circle. Suppose that the lines AB and DC intersect at P and the lines AD and BC intersect at Q. From Q, draw the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

Problem 13. (IMO proposal) Circles $\omega, \omega_1, \omega_2$ are externally tangent to each other in points $C = \omega \cap \omega_1$, $E = \omega_1 \cap \omega_2$, $D = \omega_2 \cap \omega$. Lines ℓ_1 and ℓ_2 are parallel and such that ℓ_1 is tangent to ω and ω_1 at points G and A, respectively, and ℓ_2 is tangent to ω and ω_2 at points F and B, respectively. Prove that AD and BC intersect in the circumcenter of $\triangle CDE$.

3. More Problems

Problem 14. Let ABC be a triangle. A line parallel to BC intersects the lines AB and AC at D and E, respectively. Let P be a point inside the triangle ADE, and let F and G be the intersection points of DE with BP and CP, respectively. Show that A lies on the radical axis of the circumcircles of PDG and PFE.

Problem 15. Let BB_1, CC_1 be altitudes of the triangle ABC, and let H be their intersection point. Assume $AB \neq AC$. Let M be the midpoint of BC, and D be the intersection of the lines BC and B_1C_1. Prove that DH is perpendicular to AM.