Quadratic Functions and Recursion

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Recall that the discriminant of a quadratic polynomial $ax^2 + bx + c$ is $\Delta = b^2 - 4ac$. The roots of

$$ax^2 + bx + c = 0$$

are given by the formulas

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}, \ x_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

1. Viète's relations

Problem 1. Let $E(x) = \frac{ax^2+bx+c}{x^2+1}$. Prove that if x_1 and x_2 are the roots of the equation

$$bx^2 - (c-a)x - b = 0,$$

then $E(x_1) + E(x_2) = a + c$.
Solution: We have $x_1 + x_2 = \frac{c-a}{b}$ and $x_1 x_2 = -1$. Thus $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = \frac{(c-a)^2}{b^2} + 2$. Computing we obtain

$$E(x_1) + E(x_2) = \frac{ax_1^2 + bx_1 + c}{x_1^2 + 1} + \frac{ax_2^2 + bx_2 + c}{x_2^2 + 1}$$
$$= \frac{2a(x_1x_2)^2 + (a+c)(x_1^2 + x_2^2) + bx_1x_2(x_1 + x_2) + b(x_1 + x_2) + 2c}{(x_1x_2)^2 + (x_1 + x_2)^2 + 1} = a + c.$$

Problem 2. Let $M = \{x \in \mathbb{R}\,,\, x^2 - ax + b = 0\}$ and $N = \{x \in \mathbb{R}\,,\, x^2 - bx + c = 0\}$. Find a, b, c knowing that $M \cup N = \{1, a, b, c\}$.
Solution: Let S and P be the product of the elements in $M \cup N$. On the one hand $S = a + b + c + 1$ and $P = abc$. On the other hand, By Viète's relations, $S = a + b$ and $P = bc$. Hence $c + 1 = 0$ and $abc = bc$. Thus $c = -1$. If $b \neq 0$, then $a = 1$. But then 1 is the root of either $x^2 - x + b = 0$, or of $x^2 - bx + 1 = 0$, so either $1 - 1 + b = 0$ or $1 - b + 1 = 0$, neither being possible.

Thus $b = 0$. In that case the roots of the first equation are a and 0 and of the second 1 and -1. To avoid repetitions, we should impose $a \neq 0, 1, -1$.

Problem 3. Given that x_1 and x_2 are the roots of the equation $x^2+ax+b = 0$, find a quadratic equation with roots

$$y_1 = \frac{m}{x_1} + \frac{n}{x_2}, \quad y_2 = \frac{n}{x_1} + \frac{m}{x_2}$$

where m and n are some arbitrary real numbers.
Solution: We need to express $y_1 + y_2$ and $y_1 y_2$ in terms of a and b using Viète's relations. After performing the computations we obtain the equation

$$b^2y^2 + ab(m+n)y + mna^2 + b(m-n)^2 = 0.$$

Problem 4. Let a and b be positive integers such that $a^2 + b^2$ is a prime number. Prove that the equation $x^2 + ax + b + 1 = 0$ does not have integer roots.
Solution: $a^2+b^2 = (x_1+x_2)^2+(x_1x_2-1)^2 = (x_1^2+1)(x_2^2+1)$. If one root is an integer, so is the other, and they are nonzero. Hence a^2+b^2 is composite, a contradiction.

Problem 5. Find all positive integers a, b, c such that the equations

$$x^2 - ax + b = 0, \quad x^2 - bx + c = 0, \quad x^2 - cx + a = 0$$

have integer roots.
Solution: The roots must also be positive. Write $x_1 + x_2 = a = x_5x_6$, $x_3 + x_4 = b = x_1x_2$, $x_5 + x_6 = c = x_3x_4$. Adding we obtain

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = x_1x_2 + x_3x_4 + x_5x_6.$$

This is equivalent to

$$(x_1 - 1)(x_2 - 1) + (x_3 - 1)(x_4 - 1) + (x_5 - 1)(x_6 - 1) = 3.$$

On the left there are only non-negative integers, so they can be $0, 0, 3$; $0, 1, 2$; $1, 1, 1$. A case-check gives the solution.

Problem 6. Let x_1, x_2 be the roots of the equation $x^2 + ax + bc = 0$ and x_2, x_3 be the roots of the equation $x^2 + bx + ac = 0$ with $ac \neq bc$. Show that x_1, x_3 are the roots of the equation $x^2 + cx + ab = 0$.

Solution: We have that $x_1x_2 = bc, x_2x_3 = ac, x_1 + x_2 = -a, x_2 + x_3 = -b$. These together with $ac \neq bc$ imply $x_2 = c$ and $x_1 + x_3 = -c, x_1x_3 = ab$.

Problem 7. Find all real solutions of the equation

$$\sqrt[4]{97 - x} + \sqrt[4]{x} = 5$$

Solution: Let $y = \sqrt[4]{97 - x}$ and $z = \sqrt[4]{x}$. Then $y + z = 5$ and $y^4 + z^4 = 97$. We set $P = yz$ and $S = y + z$. Then solving the system we obtain $S = 5$, $S^4 - 4s^2P + 2S^2 = 97$. Therefore $P = 6$ or $P = 44$. For $(S, P) = (5, 6)$ we find $(y, z) = (2, 3)$ or $(y, z) = (3, 2)$. These lead to two solutions: $x = 16$ and $x = 81$. In the case $(S, P) = (5, 44)$ there are no real solutions.

2. Second order linear recursive sequences

Problem 8. A newly-born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. How many pairs will there be in two years?
Solution: The number of rabbits after n months is the Fibonacci number F_n. It satisfies the recursive relation $F_0 = 0, F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$. Solving we get

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Problem 9. Find the general term of the sequence $x_0 = 6$, $x_1 = 20$, and

$$(n+1)(n+2)x_n = 5(n+1)(n+3)x_{n-1} - 6(n+2)(n+3)x_{n-2}, n \geq 2.$$

Solution: Dividing by $(n+1)(n+2)(n+3)$ we obtain

$$\frac{x_n}{n+3} = 5\frac{x_{n-1}}{n+2} - 6\frac{x_{n-2}}{n+1}.$$

Setting $y_n = \frac{x_n}{n+3}$, we have $y_0 = 2$, $y_1 = 5$, $y_{n+1} = 5y_n - 6y_{n-1}$. So $y_n = 2^n + 3^n$, and $x_n = (n+3)(2^n + 3^n)$.

Problem 10. In how many ways can one tile a $2n \times 3$ rectangle with 2×1 tiles?
Solution: Denote by u_n the number of such tilings. Start tiling the rectangle from the short side of length 3. With a case-by-case reasoning we obtain

$$u_{n+1} = 3u_n + 2v_n,$$

where v_n is the number of tilings of a $(2n-1) \times 3$ rectangle with a 1×1 square missing in one corner. Looking at the possible tilings of a $(2n-1) \times 3$ rectangle with a 1×1 corner removed we find

$$v_{n+1} = u_n + v_n.$$

Substituting we obtain $u_{n+1} = 4u_n - u_{n-1}$. The characteristic equation is

$$\lambda^2 - 4\lambda + 1 = 0.$$

Its roots are $\lambda_{1,2} = 2 \pm \sqrt{3}$. We compute easily $u_1 = 3$ and $v_1 = 1$, so $u_2 = 3 \cdot 3 + 2 \cdot 1 = 11$. The desired general term formula is then

$$u_n = \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)(2+\sqrt{3})^n + (\sqrt{3}-1)(2-\sqrt{3})^n].$$

Problem 11. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$ and

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), \quad n = 1, 2, 3, \ldots,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.
Solution: Denote the vertices of the hexagon $A_1 = A, A_2, A_3, A_4, A_5 = E, A_6, A_7, A_8$ in a successive order. Any time the frog jumps back and forth, it makes two jumps. Hence, to get from A_1 to any vertex with odd index, in particular to A_5, it makes an even number of jumps. This shows that $a_{2n-1} = 0$.

We compute the number of paths with $2n$ jumps recursively. Consider the case $n > 2$. After two jumps, the frog ends at A_1, A_3 or A_7. It can end at A_1 via A_2 or A_8. Also, the configurations where it ends at A_3 or A_7 are symmetric, so they can be treated simultaneously. If we denote by b_{2n} the number of ways of getting from A_3 to A_5 in $2n$ steps, we obtain the recursion $a_{2n} = 2a_{2n-2} + 2b_{2n-2}$. On the other hand, if the frog starts at A_3, then it can either return to A_3 in two steps (which can happen in two different ways), or end at A_1 (here it is important that $n > 2$). Thus we can write $b_{2n} = a_{2n-2} + 2b_{2n-2}$.

The characteristic equation is $\lambda^2 - 4\lambda + 2 = 0$, with roots $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

Consequently there exist constants α, β, determined by the initial condition, such that $a_{2n} = \alpha x^{n-1} + \beta y^{n-1}$. To determine α and β, note that $a_2 = 0$, $b_2 = 1$, and using the recursive relation $a_4 = 2$ and $b_4 = 3$. We obtain $\alpha = 1/\sqrt{2}$ and $\beta = -1/\sqrt{2}$, whence

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), \quad \text{for } n \geq 1.$$