Symmetric Polynomials

Dimitar Grantcharov, Mid-Cities Math Circle

For any $1 \leq r \leq n$ let

$$S_r(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_r}$$

be the r^{th} **elementary symmetric polynomial** in $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Viète's Relations: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, then for each $1 \leq r \leq n$:

$$S_r(\alpha_1, \alpha_2, \ldots, \alpha_n) = (-1)^r \frac{a_{n-r}}{a_n}$$

Any symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be expressed as a polynomial in the elementary symmetric polynomials in $\alpha_1, \alpha_2, \ldots, \alpha_n$.
Example: $\sum_{1 \leq i \neq j \leq n} \alpha_i^3 \alpha_j = S_1^2 S_2 - 2S_2^2 - S_1 S_3 + 4S_4$ where $S_k = S_k(\alpha_1, ..., \alpha_n)$.

For the r^{th} **power sum** $P_r(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^n \alpha_i^r$, for any positive integer r, we can get an expression as a polynomial in the elementary symmetric polynomials by repeatedly using:
Newton's Identities: For each $1 \leq r \leq n$

$$P_r - S_1 P_{r-1} + S_2 P_{r-2} - \ldots + (-1)^{r-1} S_{r-1} P_1 + (-1)^r S_r \cdot r = 0$$

and for each $r > n$:

$$P_r - S_1 P_{r-1} + S_2 P_{r-2} - \ldots + (-1)^n - 1 S_{n-1} P_{n-r+1} + (-1)^n S_n P_{r-n} = 0$$

where P_k and S_k stand for the k^{th} power sum and the k^{th} elementary symmetric polynomial of $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively.

Newton's identities for $r > n$ can be proven easily using Viète's relations: $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the polynomial $x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^{n-1} S_{n-1} x + (-1)^n S_n$, so for each $1 \leq i \leq n$:

$$\alpha_i{}^r - S_1 \alpha_i{}^{r-1} + S_2 \alpha_i{}^{r-2} - \ldots + (-1)^{n-1} S_{n-1} \alpha_i{}^{r-n+1} + (-1)^n S_n \alpha_i{}^{r-n} =$$

$= {\alpha_i}^{r-n}\left[{\alpha_i}^n - S_1{\alpha_i}^{n-1} + S_2{\alpha_i}^{n-2} - \ldots + (-1)^{n-1}S_{n-1}\alpha_i + (-1)^n S_n\right] = 0$

Summing up for $1 \le i \le n$, we get

$$P_r - S_1P_{r-1} + S_2P_{r-2} - \ldots + (-1)^{n-1}S_{n-1}P_{n-r+1} + (-1)^n S_n P_{r-n} = 0$$

as desired.

Problem 1. Find the zeros of the polynomial

$$P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$$

knowing that the sum of two of them is 4.

Solution: Denote the zeros of $P(x)$ by x_1, x_2, x_3, x_4 and assume that $x_1 + x_2 = 4$. The first Viète relation gives $x_1+x_2+x_3+x_4 = 6$, hence $x_3+x_4 = 2$. The second Viète relation can be written as

$$x_1x_2 + x_3x_4 + (x_1 + x_2)(x_3 + x_4) = 18,$$

from where we deduce that $x_1x_2 + x_3x_4 = 18 - 2 \cdot 4 = 10$. This, combined with the fourth Viète relation $x_1x_2x_3x_4 = 25$ shows that the products x_1x_2 and x_3x_4 are roots of the quadratic equation $u^2 - 10u + 25 = 0$. Hence $x_1x_2 = x_3x_4 = 5$, and therefore x_1 and x_2 satisfy the quadratic equation $x^2-4x+5 = 0$, while x_3 and x_4 satisfy the quadratic equation $x^2-2x+5 = 0$. We conclude that the zeros of $P(x)$ are $2+i, 2-i, 1+2i, 1-2i$.

Problem 2. Let $P(x)$ be a monic polynomial of third degree with integer coefficients and $P_1(x)$ the monic polynomial with zeros equal to the squares of the zeros of $P(x)$. Prove that $P_1(x)$ also has integer coefficients.

Solution: This follows from the equalities

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)$$
$$x_1^2x_2^2 + x_1^2x_3^3 + x_2^2x_3^2 = (x_1x_2 + x_1x_3 + x_2x_3)^2 - 2x_1x_2x_3(x_1 + x_2 + x_3)$$
$$x_1^2x_2^2x_3^2 = (x_1x_2x_3)^2.$$

Problem 3. Let a, b, c be real numbers. Show that $a \ge 0$, $b \ge 0$ and $c \ge 0$ if and only if $a + b + c \ge 0$, $ab + bc + ca \ge 0$, and $abc \ge 0$.

Solution: If $a \ge 0$, $b \ge 0$, $c \ge 0$, then obviously $a+b+c > 0$, $ab+bc+ca \ge 0$, and $abc \ge 0$. For the converse, let $u = a + b + c$, $v = ab + bc + ca$, and

$w = abc$, which are assumed to be positive. Then a, b, c are the three zeros of the polynomial

$$P(x) = x^3 - ux^2 + vx - w.$$

Note that if $t < 0$, that is if $t = -s$ with $s > 0$, then $P(t) = s^3 + us^2 + vs + w > 0$, hence t is not a zero of $P(x)$. It follows that the three zeros of $P(x)$ are nonnegative, and we are done.

Problem 4. If $x + y + z = 0$, prove that

$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^5 + y^5 + z^5}{5} = \frac{x^7 + y^7 + z^7}{7}.$$

Solution: Consider the polynomial $P(X) = X^3 + pX + q$, whose zeros are x, y, z. Then

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + xz + yz) = -2p.$$

Adding the relations $x^3 = -px - q$, $y^3 = -py - q$, and $z^3 = -pz - q$, which hold because x, y, z are zeros of $P(X)$, we find

$$x^3 + y^3 + z^3 = -3q.$$

Similarly

$$x^4 + y^4 + z^4 = -p(x^2 + y^2 + z^2) - q(x + y + z) = 2p^2,$$

and therefore

$$x^5 + y^5 + z^5 = -p(x^3 + y^3 + z^3) - q(x^2 + y^2 + z^2) = 5pq,$$
$$x^7 + y^7 + z^7 = -p(x^5 + y^5 + z^5) - q(x^4 + y^4 + z^4) = -5p^2q - 2p^2q = -7p^2q.$$

The relation from the statement reduces to the obvious

$$\frac{-2p}{2} \cdot \frac{5pq}{5} = \frac{-7p^2q}{7}.$$

Problem 5. Solve the system

$$x + y + z = 1$$
$$xyz = 1,$$

knowing that x, y, z are complex numbers of absolute value equal to 1.
Solution: Taking the conjugate of the first equation we obtain

$$\overline{x} + \overline{y} + \overline{z} = 1,$$

hence

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Combining this with $xyz = 1$ we obtain

$$xy + yz + xz = 1.$$

Therefore x, y, z are the roots of the polynomial equation

$$t^3 - t^2 + t - 1 = 0,$$

which are $1, i, -i$. Any permutation of these three complex numbers is a solution to the original system of equations.

Problem 6. Solve the system

$$\begin{aligned} x + y + z &= 5 \\ x(y+z)^2 + y(x+z)^2 + z(x+y)^2 &= -14 \\ x^2(y+z) + y^2(x+z) + z^2(x+y) &= 34. \end{aligned}$$

Solution: Adding the second and the third equation we obtain

$$2(x+y+z)(xy+yz+zx) = 20.$$

Hence $xy + yz + zx = 2$. From the second equation we obtain $xyz = -8$. It follows that x, y, z are the roots of the equation

$$t^3 - 5t^2 + 2t + 8 = 0.$$

So $\{x, y, z\} = \{-1, 2, 4\}$.

Problem 7. Solve the system of equations

$$\begin{aligned} x + y + z &= 5 \\ \frac{x}{zy} + \frac{y}{zx} + \frac{z}{xy} &= \frac{9}{4} \\ x^3 + y^3 + z^3 - 3xyz &= 5. \end{aligned}$$

Solution: Using the identity

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)$$

and the first equation, we obtain from the third equation that

$$x^2+y^2+z^2-xy-xz-yz=1.$$

Squaring the first equation and subtracting this we obtain

$$xy+xz+yz=8.$$

Adding twice this relation to the second equation and using the first equation we obtain $xyz=4$. Thus x,y,z are the roots of the polynomial equation

$$t^3-5t^2+8t-4=0.$$

The roots are $t=1,2,2$, hence x,y,z are a permutation of these numbers.

Problem 8. Solve the system of equations

$$\begin{aligned} x+y+z&=4\\ x^2+y^2+z^2&=14\\ x^3+y^3+z^3&=34. \end{aligned}$$

Solution: Let x,y,z be the roots of the equation $t^3-at^2+bt-c=0$. From the first equation we obtain $a=4$, from the first two $b=1$. Also $P_3-aP_2+bP_1-cP_0=0$ yields $34-4\cdot 14+1\cdot 4-3c=0$, hence $c=-6$. We deduce that x,y,z are the roots of the equation $t^3-4t^2+t+6=0$. The roots are $1, \frac{5\pm\sqrt{21}}{2}$.

Problem 9. Find all solutions (including complex) to the system of equations

$$\begin{aligned} x+\frac{1}{2}y+\frac{1}{2}z&=1\\ 2x^2+\frac{1}{2}y^2+\frac{1}{2}z^2&=1\\ 4x^3+\frac{1}{2}y^3+\frac{1}{2}z^3&=4. \end{aligned}$$

Solution: Let us rewrite the system as

$$\begin{aligned}(2x)+y+z&=2\\(2x)^2+y^2+z^2&=2\\(2x)^3+y^3+z^3&=8.\end{aligned}$$

Set $2x=x_1$ and consider the polynomial $P(X)=X^3+pX^2+qX+r$ with roots x_1,y,z. Then

$$q=\frac{1}{2}(x_1+y+z)^2-\frac{1}{2}(x_1^2+y^2+z^2)=1.$$

Also

$$x_1^3+y^3+z^3-2p+2q-3r=0$$

so $3r=8-4+2=6$. Thus $r=2$. We have $P(X)=X^3-2X^2+X-2$. The roots are $2,i,-i$. Thus the solutions to the original system are:
$(1,i,-i)$, $(1,-i,i)$, $(i/2,2,-i)$, $(i/2,-i,2)$, $(-i/2,2,i)$, $(-i/2,i,2)$.

Problem 10. Find all real numbers r for which there is at least a triple (x,y,z) of nonzero real numbers such that

$$x^2y+y^2z+z^2x=xy^2+yz^2+zx^2=rxyz.$$

Solution: Dividing by the nonzero xyz yields $x/z+y/x+z/y=y/z+z/x+x/y=r$. Let $a=x/y$, $b=y/z$, $c=z/x$. Then $abc=1$, $1/a+1/b+1/c=r$, $a+b+c=r$. Hence

$$\begin{aligned}a+b+c&=r\\ab+bc+ca&=r\\abc&=1.\end{aligned}$$

We deduce that a,b,c are the solutions of the polynomial equation $t^3-rt^2+rt-1=0$. This equation can be written as

$$(t-1)[t^2-(r-1)t+1]=0.$$

Since it has three real solutions, the discriminant of the quadratic must be positive. This means that $(r-1)^2-4\geq 0$, leading to $r\in(-\infty,-1]\cup[3,\infty)$. Conversely, all such r work.