AUDIENCE: Teachers (middle school teachers in particular)

DIFFICULTY: Moderate/Challening

PREREQUISITES: algebra

MATH TOPICS: combinatorics, fundamental principal of counting, enumeration

PUBLIC ABSTRACT (Draft): Who would have thought parking your car could be expressed as an algebraic function? We are going to take a look at a problem that deals with parking cars in a one-way parking lot. Each driver in a line of cars will choose which parking spot they prefer to park in. The list of preferred parking spots is called a parking sequence. When they come to their desired parking spot, if it is open they can park there, if it is not open, they will park in the next open spot. Not all parking sequences result in all cars ending in a parking spot. There are many arrangements that do result in all cars in a parking spot. We will work to find how many of these sequences exist. We will also consider the scenario where we have a round-a-bout parking lot. You'll never think of parking your car in the same way!

DISCUSSION (Generalization):

We have a line of n parking spots on the right side of a one way street. The sequence $(p_1, p_2, ..., p_n)$ is the list of preferred parking spots for n cars. p_1 is the first car's preferred spot, p_2 is the second cars preferred spot, and so on.

As the cars go down the street in increasing order, they park in their preferred spot if it is available. If their preferred spot is already taken, they take the next available spot. They cannot turn around, back up, or go around the block to try again.

Since there are n cars and each car has n choices for a preferred parking spot, using the Fundamental Counting Principle (multiplicative rule), the total number of parking sequences is n^n.

A parking sequence is *successful* if all n cars find a parking spot.

Suppose we have 4 cars and 4 parking spaces. The parking sequence $(1, 1, 1, 1)$ means that all 4 drivers prefer to park in the first parking space. This is a successful parking sequence.

- The first driver parks in the first parking spaces.
- The second driver wants to park in the first parking space, but it is occupied. Following the rules for parking, the second driver will park in the second space.
- In a similar fashion, the third driver will park in the third space and the fourth driver will park in the fourth space.

Consider the parking sequence $(2, 1, 3, 2)$.

- The first driver will park in the second space.
- The second driver will park in the first space.
- The third driver will park in the third space.
- The fourth driver wants to park in the second space but it is occupied. The fourth driver cannot park in the third space because it is also occupied. The fourth driver can park in the fourth space.

Since all drivers are parked in a parking space following the rules, $(2, 1, 3, 2)$ is a successful parking sequence.

The parking sequence $(4, 4, 1, 1)$ is not a successful parking sequence. Since two drivers prefer the fourth parking space, the second driver will not be able to park without turning around, backing up, or trying again by going around the block.

After experimenting with several specific examples for the number of cars, develop a conjecture for the number of successful sequences. What are some rules for a parking sequence to be successful or not?

- If all drivers prefer a different spot, then we have a successful parking sequence. So the number of successful parking sequences is at least $n!$.
- If no cars prefer the first parking spot, then we do not have a successful parking sequence. If no one prefers the first parking space, this means each of the n drivers only has $(n-1)$ choices, i.e. there are $(n-1)^n$ parking sequences that fall into this category and are unsuccessful. This leads to an upper bound for the number of successful parking sequences of $n^n - (n-1)^n$.
- We know that no more than one driver can prefer the n^{th} parking space. How does this affect the bounds?

You may be able to list all successful parking sequences for $n = 1, 2, 3$, and 4. The answer is quite large for $n > 4$. Thus, we introduce a new scenario.

Round-a-bout scenario: we have a round-a-bout (meaning the parking spaces are arranged in a circle) with $(n + 1)$ parking spots on the right side and we label the spots from when you enter the round-a-bout (I.e. cul-de-sac). There are still only n cars and $(p_1, p_2, ..., p_n)$ is the list of preferred parking spots for n cars. p_1 is the first car's preferred spot, p_2 is the second cars preferred spot, and so on.

As the cars go around the round-a-bout in increasing order, they park in their preferred spot if it is available. If their preferred spot is already taken, they take the next available spot and they can go around the circle to park in the next available spot. They cannot turn around or back up.

All parking sequences in this scenario are successful because the cars are able to "try again" by going back around the circle; there are $(n+1)^n$ of these parking sequences; there are n cars and each driver has $(n+1)$ choices for a preferred parking spot. How do these parking sequences connect to the original problem?

Connections between the Round-a-bout Scenario and the original problem: If we have a parking sequence in the Round-a-bout Scenario that leaves the $(n+1)^{\text{st}}$ parking spot empty, then this would also be a successful parking sequence in the original scenario with the one-way street. So what is the probability that a Round-a-bout parking sequence would leave the $(n+1)^{\text{st}}$ parking spot empty?

Observations in the Round-a-bout Scenario: If $S = (p_1, p_2, ..., p_n)$ is a parking sequence that leaves the k^{th} parking spot empty, then $S + \mathbf{1} = (p_1 + 1, p_2 + 1, ..., p_n + 1)$ will leave the $(k+1)^{\text{st}}$ parking space empty. This means that we can partition the Round-a-bout Scenario parking sequences into $(n+1)$ groups, where each group represents the parking space that is left empty. Thus, the number of parking sequences in one group is $\frac{(n+1)^n}{(n+1)}$ since there were $(n+1)^n$ Round-a-bout Scenario parking sequences and $(n+1)$ parking spaces that could be left empty. Thus, the number of parking sequences in the Round-a-bout Scenario that leave the $(n+1)^{\text{st}}$ parking space empty is $(n+1)^{(n-1)}$.

So for the original problem with n cars choosing any one of n parking spaces on a one way street, the number of successful parking sequences is $(n+1)^{(n-1)}$.

HANDOUT below

Sources:

Beck, Matthais (October 2010). Parking Functions. *Stanford Math Circle.* Retrieved April 10, 2012, from `http://math.stanford.edu/circle/2010fall.php`

Johnston, Elgin. Where Can I Park? *Math Teachers' Circle Network.* Retrieved April 10, 2012, from `http://mathteacherscircle.org/resources/materials/IJohnstonParking%20Function.pdf`

DISCUSSION:

Suppose we have 4 empty parking spots along the right side of a dead end street and there are 4 people with cars that need to park there and there are no parking spots on the other side of the street. The parking spots are numbered 1, 2, 3, and 4 with the farthest spot being 4.

enter $\rightarrow$	1	2	3	4	Dead End

Each of the 4 drivers has a preference where they would like to park. A listing of their respective preferred parking spots is called a parking sequence. Some example parking sequences are $(1, 2, 3, 4), (2, 3, 1, 2), (4, 2, 3, 1)$, and $(1, 4, 4, 4)$. The parking sequence $(2, 3, 1, 2)$ means that the first driver prefers the 2$^{\text{nd}}$ parking spot, the second driver prefers the 3$^{\text{rd}}$ parking spot, the third driver prefers the 1$^{\text{st}}$ parking spot, and the fourth driver prefers the 2$^{\text{nd}}$ parking spot. As this example illustrates, it is possible that two (or more) different drivers prefer the same parking spot.

Try This 1: *How many parking sequences are exist?*

Given a parking sequence (p_1, p_2, p_3, p_4), the following rule is used to "park" the cars. As car i enters the street, they will park in their preferred spot, p_i, if the parking spot is empty. If the parking spot is occupied, then they will park in the next empty parking spot that they come to. A parking sequence is considered successful if each driver can find a parking spot without needing to turn around or reverse to try to find an available place.

The parking sequences $(1, 2, 3, 4), (2, 3, 1, 2)$, and $(4, 2, 3, 1)$, are all successful parking sequences. For example, for the parking sequence $(2, 3, 1, 2)$ the first car will park in spot 2, the second car will park in spot 3, and the third car will park in spot 1. When the fourth car goes to park, this driver's preferred spot, 2, is occupied. So they drive past and take the next available spot; since spot 3 is also occupied, the fourth car will park in parking place 4.

The parking sequence $(1, 4, 4, 4)$ is not a successful parking sequence. The first car will park in spot 1 and the second car will park in spot 4. When the third car tries to park, this driver's preferred spot, 4, is occupied. They drive past, but there are no available spots beyond parking spot 4. The only way they could park is if they reversed back to an open spot. This is not permitted for a parking sequence to be considered successful.

Try This 2: *List several other parking sequences and sort them according to whether or not they are successful. What are some conditions for a parking sequence to be successful or not?*

Try This 3: *Repeat the process above for each of the cases* $n = 1$, $n = 2$ *and* $n = 3$. *What patterns arise from these examples?*

Try This 4: *Lets look at a larger case, say $n = 10$ parking places and people.*

A. *How many people can/must prefer the 1st parking spot in a successful parking sequence? Could no one prefer the 1st parking spot in a successful parking sequence? Could everyone prefer the first parking spot in a successful parking sequence?*

B. *How many people can/must prefer the 10th parking spot in a successful parking sequence? Could not one prefer the 10th parking spot in a successful parking sequence? Could everyone prefer the 10th parking spot in a successful parking sequence?*

C. *How many people can prefer either spots 8, 9 or 10 in a successful parking sequence? Can we extend these ideas to general rules about successful parking sequences?*

Try This 5: *Suppose you have a successful parking sequence. If the cars return home at the end of the day in a different order, do we still have a successful parking sequence?*

Try This 6: *Let* $(p_1, p_2, p_3, \ldots, p_n)$ *be a parking sequence. Take the numbers in the sequence and rearrange them into nondecreasing order:* $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ *(so the* a_j*'s are simply the* p_k*'s in a different order). What do you notice about the number* a_k *in position* k *for successful parking sequences?*

Round-a-bout Scenario: Changing the problem a little, now assume we have $(n+1)$ parking spots on a one-way round-a-bout. We still have n cars that need to find a parking spot and each driver has a preferred parking place and $(p_1, p_2, p_3, \ldots, p_n)$ is a parking sequence. As car i enters the round-a-bout, they will park in their preferred spot, p_i, if the parking spot is empty. If the parking spot is occupied, then they will park in the next empty parking spot that they come to. In this scenario if all spots past p_i are occupied, the driver can continue around the round-a-bout until they find an empty space. A parking sequence is considered successful if each driver can find a parking spot. For the round-a-bout scenario, all parking sequences are successful.

Try This 7: *How many (successful) parking sequences exist for the round-a-bout scenario?*

Try This 8: *What is the connection between the parking sequences for the round-a-bout scenario and the successful parking sequences for the original problem (dead end street)?*

Every parking sequence in the round-a-bout scenario results in exactly one parking spot remaining empty after all n cars have parked. In fact, if we know that $S = (p_1, p_2, ..., p_n)$ is a parking sequence that leaves the k^{th} parking spot empty, then $S + 1 = (p_1 + 1, p_2 + 1, ..., p_n + 1)$ will leave the $(k+1)^{\text{st}}$ parking spot empty.

Try This 9: *How many parking sequence in the round-a-bout scenario leave spot 1 empty? spot 2 empty? .... spot* $(n + 1)$ *empty?*

Try This 10: *How many parking sequences are successful for a dead end street with* n *parking spaces and* n *cars needing to park?*