Mathematical Origami:
PHiZZ Dodecahedron

We will describe how to make a *regular dodecahedron* using Tom Hull's *PHiZZ* modular origami units. First we need to know how many faces, edges and vertices a dodecahedron has. Let's begin by discussing the Platonic solids.

The Platonic Solids

A *Platonic solid* is a convex polyhedron with congruent regular polygon faces and the same number of faces meeting at each vertex. There are five Platonic solids: *tetrahedron, cube, octahedron, dodecahedron,* and *icosahedron*.

The solids are named after the ancient Greek philosopher Plato who equated them with the four classical elements: earth with the cube, air with the octahedron, water with the icosahedron, and fire with the tetrahedron). The fifth solid, the dodecahedron, was believed to be used to make the heavens.

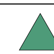
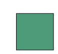
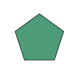
Why Are There Exactly Five Platonic Solids? Let's consider the vertex of a Platonic solid. Recall that the same number of faces meet at each vertex. Then the following must be true.

- There are at least three faces meeting at each vertex.
- The sum of the angles at a vertex must be less than 360 degrees. (Otherwise we would not have a convex polyhedron.)
- It follows that each interior angle of a polygon face must measure less than 120 degrees.
- The only regular polygons with interior angles measuring less than 120 degrees are the equilateral triangle, square and regular pentagon.

Therefore each vertex of a Platonic solid may be composed of

- 3 equilateral triangles (angle sum: $3 \times 60° = 180°$)
- 4 equilateral triangles (angle sum: $4 \times 60° = 240°$)
- 5 equilateral triangles (angle sum: $5 \times 60° = 300°$)
- 3 squares (angle sum: $3 \times 90° = 270°$)
- 3 regular pentagons (angle sum: $3 \times 108° = 324°$)

These are the five Platonic solids.

Solid	Face	# Faces/ Vertex	# Faces
tetrahedron		3	4
octahedron		4	8
icosahedron		5	20
cube		3	6
dodecahedron		3	12

How Many Vertices and Edges Does Each Solid Have?

Consider the dodecahedron. To calculate the number of vertices in a dodecahedron, we note that the solid is composed of 12 regular pentagons. If the pentagons were all separate, then we would have a total of $12 \times 5 = 60$ vertices. But 3 pentagons meet at each vertex so the number of vertices in a dodecahedron is

$$V = \frac{(\#\text{ faces}) \times (\#\text{ vertices per face})}{(\#\text{ faces per vertex})} = \frac{12 \times 5}{3} = \frac{60}{3} = 20.$$

We can calculate the number of vertices in the other Platonic solids using the same method.

To calculate the number of edges in a dodecahedron, we note that 12 regular pentagons have a total of $12 \times 5 = 60$ edges. But when we join the pentagons to make a dodecahedron, each edge meets another edge so the number of edges in a dodecahedron is

$$E = \frac{(\#\text{ faces}) \times (\#\text{ edges per face})}{2} = \frac{12 \times 5}{2} = \frac{60}{2} = 30.$$

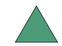
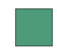
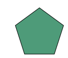
Similarly we can calculate the number of edges in the other Platonic solids.

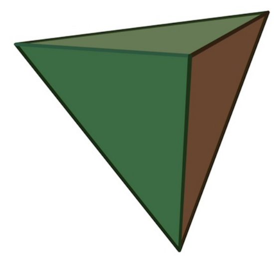
Euler's Formula for Polyhedra

We can check our answers using Euler's formula for convex polyhedra:

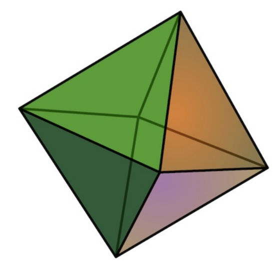
$$V - E + F = 2.$$

For each solid, the number of vertices minus the number of edges plus the number of faces equals 2.

Solid	Face	# Faces/ Vertex	# Faces	# Vertices	# Edges
tetrahedron		3	4	4	6
octahedron		4	8	6	12
icosahedron		5	20	12	30
cube		3	6	8	12
dodecahedron		3	12	20	30



tetrahedron



octahedron

Polyhedron Duals

Every Platonic solid has a *dual* polyhedron which is another Platonic solid. The dual is formed by placing a vertex in the center of each face of a Platonic solid. The resulting polyhedron is another Platonic solid. The dual of the octahedron is the cube and vice versa. Similarly the icosahedron and the dodecahedron are a dual pair. The tetrahedron is *self-dual*: its dual is another tetrahedron.

Making a PHiZZ Dodecahedron

Now that we know a dodecahedron is composed of 12 pentagon faces and a total of 30 edges, we are ready to make a dodecahedron out of PHiZZ modular origami units. Each PHiZZ unit will form one edge of the dodecahedron so we will need 30 square pieces of paper. (The 3" × 3" memo cube paper from *Staples* works well. Do not use sticky paper like Post-its.)

Refer to the *Pentagon-Hexagon Zig-Zag (PHiZZ) Unit* webpage for instructions on how to fold and join PHiZZ units.

- `http://mars.wnec.edu/~thull/phzig/phzig.html`

Video demonstrations can be found at the following links.

- PHiZZ Unit Part 1: `http://www.youtube.com/watch?v=vFYw47Wx2N8`
- PHiZZ Unit Part 2: `http://www.youtube.com/watch?v=dH-uTRdI4XU`

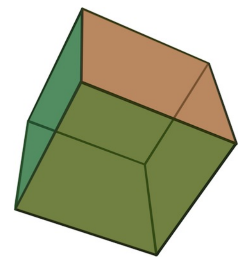
1. Begin by folding 3 PHiZZ units. Join the 3 edges to make one vertex of the dodecahedron.
2. Now add 7 more PHiZZ units to make a regular pentagon. (The extra units are needed to lock the vertices in place.)
3. Add to the pentagon until you have a complete dodecahedron.

PHiZZ units can be used to make various polyhedra composed of pentagon and hexagon faces. For example, one can make a truncated icosahedron (aka soccer ball) using 90 PHiZZ units. Try it!

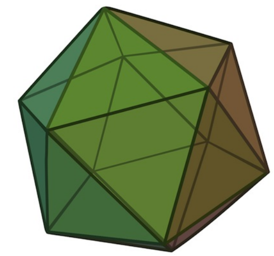
References

Hull, Tom. *Project Origami*. A K Peters Ltd, 2006.

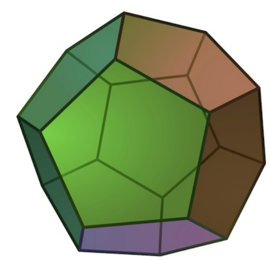
"Platonic Solid." *Wikipedia, The Free Encyclopedia.* Wikimedia Foundation, Inc. 16 February 2011.



cube



icosahedron



dodecahedron