THE REGIONS OF A CIRCLE, DIFFERENCE EQUATIONS, AND THE FIELD OF SEQUENCES

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Overview The investigation of patterns and sequences is a topic that is pervasive to many Math Circle projects. Depending on the style in which the Math Circle is conducted, the treatment of sequences can either be very informal, based on elementary hands-on examples and activities, or in the form of more traditional enrichment classes preparing students for extracurricular, competitive activities. This Math Circle project is composed of three modules that aim to provide participants with a solid mathematical introduction to the analysis of patterns based on the equally fascinating, beautiful, and useful tools consisting of generating functions, z-transforms, and the field of sequences (i.e., sequences with standard addition and multiplication defined via the Cauchy product).

The project attempts to provide this introduction in a careful and purposeful manner so as to foster the students' individual sense of discovery and creativity and in doing so further develop their mathematical confidence and maturity. To motivate the study of the field of sequences, generating functions and z-transforms, many interesting modules would be suitable (for example difference equations like generalized Fibonacci sequences $a_{n+2} = Aa_{n+1} + Ba_n$, Catalan Numbers (Dyck paths, Diagonal Triangulation of Polygons), Motzkin paths, Hipparchus problem, etc.). We chose to start our first module with the famous "Regions of a Circle" problem that leads to the analysis of the sequence

$$1, 2, 4, 8, 16, 31, 57, 99, 163, 256, \cdots .$$

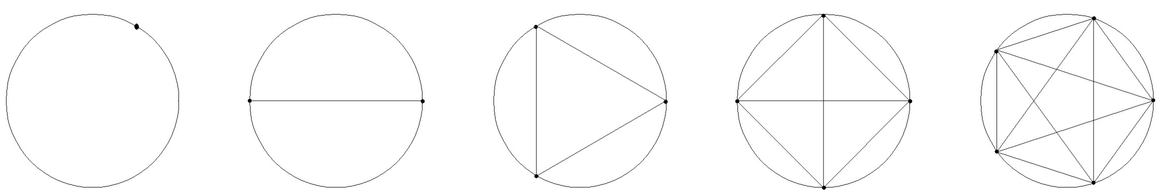
The investigation of patterns in Math Circle type settings allow for creativity, exploration, and are prime examples of problems that can and should be treated with a wide variety of mathematical tools from all fields of mathematics. That is, not only do we want students to experience the awe of finding an answer, we want them to see the beauty and mathematical power of finding answers with seemingly entirely different approaches and methods. Although some gifted students may be able to have significant success with pattern problems based solely on their own abilities and ingenuity, the purpose of our second module is to provide a template for a mathematically solid introduction to some of the standard tools that may come handy when investigating patterns: the field of sequences, generating functions and the z-transform. Finally, our third module is dedicated to explorations on how these tools can be used to give a broader base of students the ability to experience personal success with the investigations of discrete patterns and how they can be used effectively in tandem with other more geometrical and graph-theoretic methods that are often used in the study of patterns.

Regions of Circle, Part 1. This introductory module will be an open investigation of the following problem:

> *Given a circle with n points on its boundary, join all pairs of these points with straight lines. What are the maximum number of regions formed by the lines inside the circle?*

We were originally introduced to this problem from Timothy Gowers book [1] and later discovered that James Tanton had already written lesson plans for a math circle activity centered around the same problem which can be found in the lesson plans section of the *mathcircles.org* website (see http://www.mathcircles.org/node/670 or [2]). For this module, we are less interested in leading the students through a presentation similar to Tanton's to derive a formula for the number of regions using a strictly combinatorial argument (of course, there will be no complaints if gifted students are able to produce such an argument independant of guidance), but rather we are most concerned with the students developing the sequence from which we can motivate the introduction of generating functions in the next module.

A natural starting point for the circle is to simply present the regions of the circle problem, and then ask the students to produce the answers for the $n = 1, 2, 3, 4, 5$ hence leading to our first suprise.



(a) 1 region (b) 2 regions (c) 4 regions (d) 8 regions (e) 16 regions

Using this data, the question can be posed to the students, "Can anyone make a conjecture about the number of regions for $n = 6$.....for any n?" It is natural to expect that someone will make the conjecture that for any n, the number of regions created by the lines joining n pairs of dots will be 2^{n-1}. But could this conjecture be correct?

References

[1] Gowers, Timothy. *Mathematics: A Very Short Introduction*, Oxford University Press, 2002.

[2] Tanton, James. *Clip Theory, Mathematical Outpourings: Newsletters and Musing from the St. Mark's Institute of Mathematics*, 2010.