Tangle Dance Guide

Given two fairly close ropes (belts, whatever) it is easy to run this activity anywhere. It is visually very appealing and will attract crowds. It also connects to a large amount of mathematics. Many people present this as a magic trick tying a bag over the tangle. I recommend starting with the Euclidean algorithm because it is one of the most important mathematical tools from the middle school curriculum that is commonly neglected. A video of the start of a session run in this way is available at http://youtu.be/fSaI1jQbEfo .

A video following a different script by Tom Davis may be seen at:
http://www.youtube.com/watch?v=iE38AXV_dHc .

The Tom Davis notes may be found at http://www.mathcircles.org/files/tangle.pdf .

Reducing Fractions The first way students are taught to reduce fractions is to guess possible common factors. This is a good start, however at some point students should be led to discover the algorithm to reduce fractions. Any fraction in which the common factor is a moderately large to large prime is difficult to reduce by guessing because there are too many factors to try. Common factors such as 23, 43, 61, 101 will give students problems, until they learn the algorithm. Sample fractions of this type include: 529/713, 1333/2537, 1403/6161, and 3721/10007 (no one should start with this last one). You can bring these and a few more that you prepare ahead to use as examples. Of course it is not too difficult to take a moment and create one.

Starting the activity with some math Ask the audience to reduce the fraction 341/2821. The odds are they won't know how to do it. Ask them how common factors to 341/2821 compare to the common factors in 2821/341, then ask if they can write 2821/341 in a different way i.e. as a mixed fraction. They should be able to use long division to do this. Have them do it. The answer is $8\frac{93}{341}$. Remark that if the first fraction were really some number of 13ths the answer would be 8 and some number of 13ths. Thus anu common factor would also be a common factor in 93/341. Can they find it? No problem – just repeat. Consider 341/93. Write it as a mixed fraction, remove the integer part, and repeat until it is obvious how to reduce the fraction. This gives the chain:

$$341/2821 \to 2821/341 = 8\tfrac{93}{341} \to 93/341 \to 341/93 = 3\tfrac{62}{93}$$
$$\to 62/93 \to 93/62 = 1\tfrac{31}{62} \to 31/62 \to 62/31 = 2\,.$$

We conclude that 31 is a common factor to both 31 and 62, thus a common factor to 62 and 93 and *dots* thus it is a common factor to both 341 and 2821. The divisions $341/31 = 11$ and $2821/31 = 91$ then show that the original fraction is $341/2821 = 11/91$. Tell the audience that the process in the chain above is a sequence of transformations $x \mapsto 1/x$ and $x \mapsto x - 1$. Make sure that they remember why $x \mapsto 1/x$ just flips the fraction, and note that $8\frac{93}{341} \to 93/341$ is just $x \mapsto x - 1$ done eight times in a row.

Dance Time The first thing to do is to design the choreography. The two basic moves are:

T = **Twist Up** $x \mapsto x+1$

S = **Swing around** $x \mapsto -1/x$

These moves are clearly very close to the moves used to reduce fractions ($x \mapsto 1/x$ and $x \mapsto x-1$). [The names and choice of signs arise because these or standard generators of something known as the *modular group*. Other presenters use different letters, but we recommend using S and T because these are the standard names.] The choreography is a sequence of numbers starting at 0 that become more complicated as a sequence of S and T moves is applied, and then gets driven down to 0 by more moves. The presenter can start by writing a few moves down to make the fraction get a bit complicated:

$$0 \xrightarrow{T} 1 \xrightarrow{T} 2 \xrightarrow{T} 3 \xrightarrow{T} 4 \xrightarrow{T} 5 \xrightarrow{S} -1/5 \xrightarrow{T} 4/5 \cdots$$

Then take requests, "T or S next?" once the sequence is a bit longer and the fraction is something like 11/17 ask the audience what to do to drive it down to zero. They should be able to figure out a system that works.

Ask them to do the arithmetic.

Ask them to give rules to drive the fraction down to zero.

Ask them why these rules work.

Leave the choreography sequence in view.

Now get four audience volunteers for the dance. Have them stand in a square and hand each person one end from a pair of ropes so that the ropes are parallel. Write the word "UP" on paper and put it on the ground next to one person as a cheat sheet. Explain that the first move is called "Twist Up" – the person standing next to the "UP" lifts their hand and trades places with the person next to them who is not on the same rope. Have them practice this move one or two times. The next move is called "Swing Around" – everyone walks to the next corner of the square in a counter-clockwise orientation. Have your dancers practice these moves a few times. At this point have your dancers do the last move "Display" – the two people in the back hold their hands up. Now have one person let go of the rope straighten the ropes and get ready.

It is now time to dance their choreography sequence. Have the dancers display at the mid way point, then keep going and display at the end.

This is a nice trick, but it also contains many great mathematical problems and is related to many interesting mathematical applications. These rational tangles were introduced by John Conway as a tool to help organize the list of all possible knots, [1]. Knot theory is an interesting mathematical subject in its own right. Rational tangles have been used in mathematical biology to model long DNA strings, [2]. The Euclidean algorithm finds applications throughout mathematics. It will appear whenever one needs to analyze linear families of integers, continued fractions, lattices. It is used in RSA cryptography as well as many other encryption schemes, [3].

Because this activity is related to so much mathematics, it is worth exploring some of the math. The fact that the numbers come back to zero does not imply that the ropes will untangle. One needs to verify that the ropes follow the same relations. The argument that any fraction may be driven to 0 via the $S(x) = -1/x$ and $T(x) = x + 1$ moves shows that one can get to any fraction from 0 using the inverse moves $S^{-1}(x) = S(x) = -1/x$ and $T^{-1}(x) = x - 1$. Ask the audience to explain this. The same argument will show that any fraction can be obtained from 0 via a sequence of S and T moves. The *modular group* is the collection of all linear fractional transformations:

$$r(x) = \frac{ax + b}{cx + d}, \text{ such that } ad - bc = 1\,.$$

It is denoted $\mathrm{PSL}_2\mathbb{Z}$. This argument shows that the modular group is generated by the S and T transformations.

To know that the ropes follow the algebra, we need to know that the ropes satisfy the same relations that the S and T transformations satisfy. It is known that every relation between these transformations is a consequence of the relations:

$$S^2 = \mathrm{id}, \text{ and } (ST)^3 = \mathrm{id}\,.$$

One can prove that every rational tangle i.e. tangle created by the dance procedure satisfies the same relations via induction. Make the audience go through this. A good way to do this is to close a book over top of the two ropes and then do the moves in one of the relations.

> Tie a knot in one of the ropes, then tangle the result with the other rope. Can the result be rational?
>
> Show that every rational tangle has three 2-fold rotational symmetries.
>
> Is every tangle in which each rope is unknotted, that has three 2-fold rotational symmetries rational?
>
> Is there a reasonable extension of a tangle that could represent $\sqrt{2}$?

The Euclidean algorithm is the fancy name for the long division algorithm. Most students learn long division, but they do not learn the related computations involved in computing the greatest common divisor of a pair of integers, reducing a fraction, or solving linear diophantine equations such as $34x + 231y = 5$ for all integer solutions. Drilling these computations is not in the spirit of a math circle, but it would be good to include these computations as part of the regular curriculum. Here is a sample solution to a linear diophantine equation. Compare it to the rational tangle dance:

$$
\begin{aligned}
&15 \cdot 0 + 26 \cdot 1 = 26 \\
&15 \cdot 1 + 26 \cdot 0 = 15 \qquad \frac{26}{15} = 1\frac{11}{15}, \text{ so subtract } 1 \\
&15 \cdot (-1) + 26 \cdot 1 = 11 \qquad \frac{15}{11} = 1\frac{4}{11}, \text{ so subtract } 1 \\
&15 \cdot 2 + 26 \cdot (-1) = 4 \qquad \frac{11}{4} = 2\frac{3}{4}, \text{ so subtract } 2 \\
\text{note } &-1 - 2 \cdot 2 = -5 \text{ and } 1 - 2 \cdot (-1) = 3 \text{ so} \\
&15 \cdot (-5) + 26 \cdot 3 = 3 \\
&15 \cdot 7 + 26 \cdot (-4) = 1\,.
\end{aligned}
$$

References

[1] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 329–358. Pergamon, Oxford, 1970.

[2] C. Ernst and D. W. Sumners. A calculus for rational tangles: applications to DNA recombination. *Math. Proc. Cambridge Philos. Soc.*, 108(3):489–515, 1990.

[3] R. L. Rivest, A. Shamir, and L. Adleman. A method for obtaining digital signatures and public-key cryptosystems. *Comm. ACM*, 21(2):120–126, 1978.