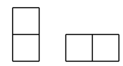
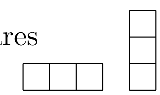
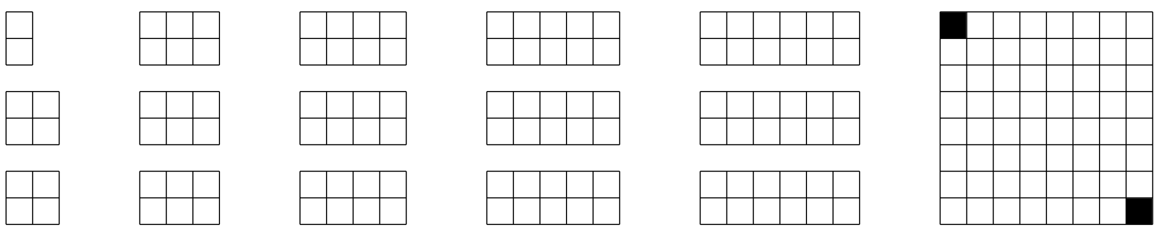
Tilings with dominos, straight, bent and triangular triominos

1. In how many different ways can a rectangular $2 \times n$ board be tiled with dominos? Start with small numbers n, then generalize.
2. Is it possible to tile a 5×5 square board with dominos?

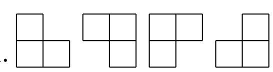


3. Is it possible to tile with dominos a 5×5 square board from which one square has been removed? Does it matter which one has been removed?
4. Is it possible to tile with dominos an 8×8 rectangular board from which two opposite corner have been removed?
5. Find all squares on an 8×8 rectangular board such that if one of these squares is removed, then the remaining part can be tiled with (straight) triominos.

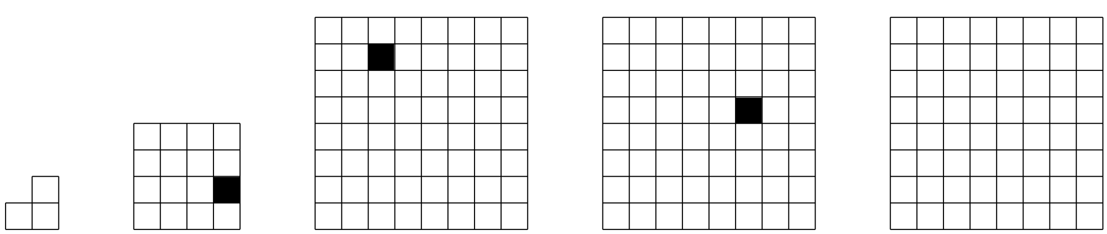


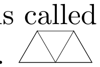


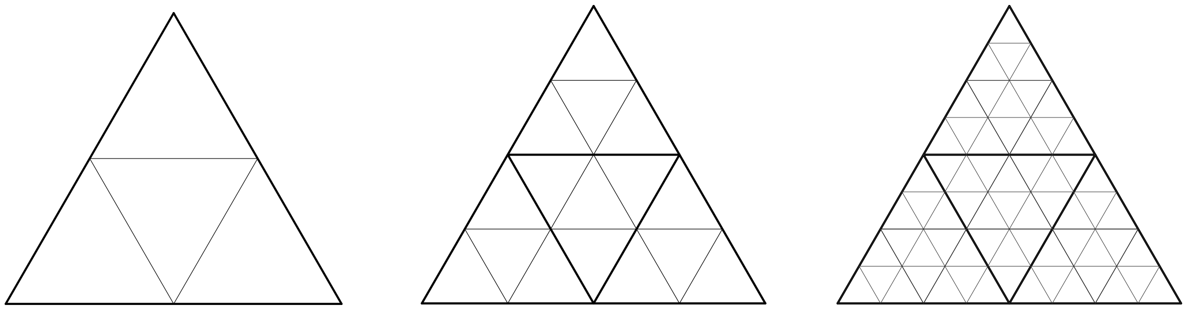
A bent triomino is of a 2×2 board from which one corner has been removed.



6. Is it possible to tile an $2^n \times 2^n$ board from which one square has been removed with bent triominos? Does it matter which square has been removed?
7. Which square boards of size $n \times n$ from which one square has been removed can be tiled with bent triominos? (Hint: The 5×5 board is special.)
8. Which $n \times m$ rectangular boards can be tiled with bent triominos?



Write $T(n)$ for a triangular board of side-length 2^n which is subdivided into equilateral triangles each of side length 1. If a triangle shares one (or two) of its sides with the large triangle, then it is called an *edge triangle*. If it shares two of its sides with the large triangle, then it is called a corner triangle. A (triangular) triomino is a *tile* consisting of three adjacent triangles. 

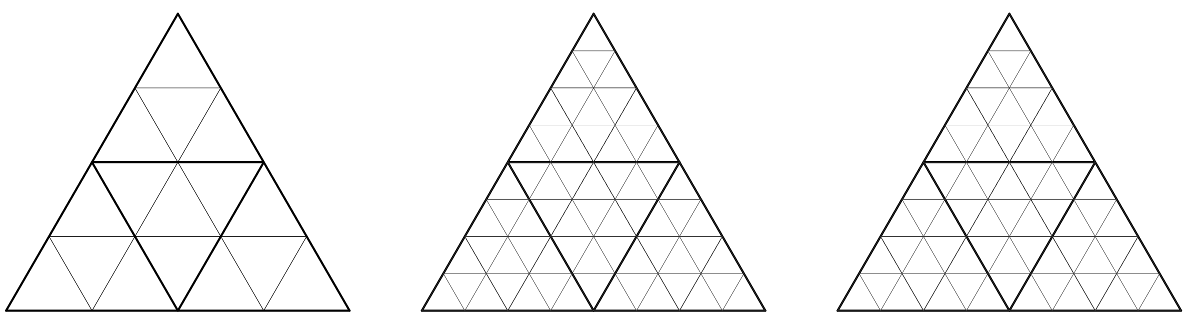


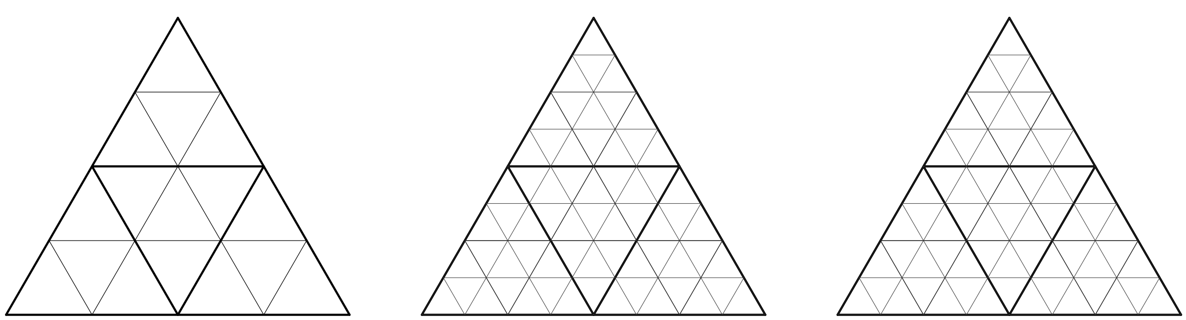
9. For which n is it possible to tile the remaining board with triangular triominos after any (one) corner triangle is removed from $T(n)$?

10. For which n is it possible to tile the remaining board with triangular triominos after any (one) edge triangle is removed from $T(n)$?

11. For which n is it possible to tile the remaining board with triangular triominos after all the corner triangles and any other (one more) triangle are removed from $T(n)$?

12. For which n is it possible to tile the remaining board with triangular triominos after if any (one) triangle not adjacent to a corner triangle is removed from $T(n)$?





Notes to the instructor

The main mathematical theme is to *use what you already know"*, to develop recursive / inductive reasoning. Much of the charm of the problems stems from tiles being very tangible, even preschoolers understanding the concept – though they may prefer manipulating physical objects where older ones just draw/indicate a tile by a short straight or broken line segment. Each of the problems selected here typically has a little twist, encouraging coming up with some new ideas out of the blue. Utilize symmetries to reduce number of cases to be considered. We plead with the instructors to NOT give too many strong clues. Popular solutions rely on parity and coloring arguments – e.g. a checker-board pattern can be the key that unlocks, do not give it away by starting with a checkerboard.
The kids love to have fun – and they SHALL have fun at a festival – they **just play** with the tiles. It is our job to nudge them to formulate questions which are almost there (often they are just not used to asking the questions), and discover that there is mathematics here which can help answer these questions in elegant and systematic ways.
Target audience: From elementary school students to college majors – the latter writing formal proofs by induction.
Equipment needed: The older the participants the less is needed. Very young one may want tangible tiles and grid.
For festival setting with many young students we use foam-board or popsicle sticks for the tiles, and poster board for drawing the grid from the craft store. Bi- or tri-color the popsicle sticks – we use masking tape and spray paint 50 at a time). Two to three inch squares or rectangles work fine for a festival setting. For older ones, or a circle setting, just use paper to sketch some grids – though, for the triangular boards it is nice to have many copies of preprinted boards of various sizes available. At the most mature level, mathematicians may sketch just a few lines that are needed to communicate the main argument – a napkin and pen are enough.

1. Typical final goal: recursion for Fibonacci numbers. E.g. knowing the answers for $n \leq 5$, how does this help to get the answer for $n = 6$.

2. While the 5×5 is special for bent triominos (see below), we found that for younger ones the 5×5 and $5 \times 5 - 1$ with dominos (!) are good starting points. Surprisingly many kids start tiling before thinking.

3. Once is is observed that 25 is odd, the instructor offers to *remove* one square (we place a rock on that square). This is a fun place to play quick games/tricks: when I do it, it works, and you just can't get it right (when I clean up the board the rock *accidentally* slipped to a different square). Finding out the pattern of which squares work and which don't can take quite a while. In the end, we usually come up with a checkerboard coloring argument, e.g. corners are black. Then the original 25 consisted of 12 white and 13 black squares. Each domino covers as many white as black squares (since we did not have a colored board, this pattern is *imagined*, but by suitably arranging the colored popsicle sticks, one may reproduce the pattern). Thus the rock better be placed on a black square .... (It remains to be shown that in any such case it is indeed possible to complete the tiling – a few cases do the job)...

5. Well known puzzle – color the board in a checker-board pattern, and each domino half white, half black. There are many variations of this question (board sizes, removing fewer or more, different size tiles, more than 2 colors).

6. One intriguing solution uses three colors to color the rectangular board along diagonals NE-SW or NW-SE. How many squares are colored in each color? Which colors does any straight trionimo cover?

7. There are (at least) two beautiful induction arguments – and students who see one often have difficulty comprehending the other in the same session – both do come up.
(1) Partition the $2^{n+1} \times 2^{n+1}$ board into four $2^n \times 2^n$ boards. The removed tile lies in one of these four. Place one bent triomino next to the center so it covers exactly one square in each of the other three smaller boards. Use *"what you know"* about the smaller boards.
(2) Four bent triominos may be *combined* to form a super bent triomino of twice the original (linear) size. Also, the moved square may be *expanded* to double its (linear) size by suitably placing one bent triomino next to it, finally aggregate groups of four squares in the original grid. Use *"what you know"* working with super-tiles.

8. Clearly, divisibility by 3 matters. Natural induction arguments go in steps of 3 (or even 6), parallel chains with separate base cases. The 5×5 board is special – why? Consider symmetries to reduce the number of cases to be considered.

9. Rectangular boards and different tiles opens up the field in many directions. Use different colored tiles that may only be places with certain orientation, or certain neighbor relations.

10. Triangular boards involve very similar arguments, but add some novelty. (Again both kinds of induction arguments appear.) Except for the obvious ones (triangles next to corners removed), almost all answers are positive. Again extensions use different tiles, colors, remove more triangles etc. Extensions to three (and higher) dimensions are encouraged.

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