The Topology of Surfaces and 3-Dimensional Spaces
Lesson Plan

August 15, 2011
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I have presented this material at math circles for students in grades 6 - 8, over the course of several meetings. Most of the material is from The Shape of Space by Jeff Weeks. A great collection of intuition-building online games are available at http://geometrygames.org/TorusGames/index.html .

1 What is Topology?

I start with a brief informal description of the difference between topology and geometry.

The properties of an object that stay the same when you bend, stretch or twist it are called the **topology** of the space. Two objects are considered the same topologically if you can deform one into the other without tearing, cutting, pinching, gluing, or other violent actions.

The properties of an object that change when you bend, stretch, or twist are the **geometry** of the space. For example, distances, angles, and curvature are parts of geometry but not topology.

I get out my play dough and make a doughnut shape. I tell the joke that a topologist is a person that can't tell a doughnut from a coffee cup, and I demonstrate. This may be easier to see with a video than with playdough, and there are some videos on youtube.

Next, I ask for a few volunteers to stand in front of the class. I start pulling out objects, one at a time, from the bags I've brought. My objects include things like: a lemon, a bunch of bananas, a milk jug (assumed to be solid), a coffee cup, a two handled vase or cup, a pretzel, a bagel, scissors, and a standard shaped torus and two-holed torus. As I pull out each object, I ask the class who should hold it, with the understanding that each volunteer hold objects with the same topology. I also introduce some vocabulary: sphere, one-holed torus, two-holed torus, three-hold torus. Last of all, I pull out the the blue object shown below, which most students initially claim is a two-holed torus (but it's not!)

2 Informal definitions

I spend a few minutes on informal definitions of dimension, boundary and size, taken from The Shape of Space *by Jeff Weeks.*

- **1-dimensional:** Only one number is required to specify a location; has length but no area. Each small piece looks like a piece of a line.
- **2-dimensional:** Two numbers are required to specify a location; has area but no volume. Each small piece looks like a piece of a plane.
- **3-dimensional:** Three numbers are required to specify a location; has volume. Each small piece looks like a piece of ordinary space.
- **n-dimensional** n numbers are required to specify a location.
- **Surface:** A 2-dimensional object (even if it isn't the surface *of* anything).
- **Boundary:** An edge of space. A traveler who reaches a boundary can go no further.
- **Finite:** Has a limited, measurable length/area/volume/
- **Infinite:** Has unlimited length/area/volume.

It is especially important that students understand how we will thinking about dimension. For example, we will consider a circle to be one-dimensional, even though we draw it wrapped around in two dimensions. The objects I pulled out of the bags are all three dimensional, but if we imagine idealized surfaces of these objects, like infinitely thin films of paint, then those surfaces would be two-dimensional. More time can be spent on these definitions, if needed, by asking students to come up with examples of one-, two-, and three-dimensional objects that are finite and infinite, with and without boundary.

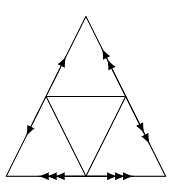
1. What is the dimension of our universe? Does it have boundary? Is it finite or infinite?

   One argument that the universe must be infinite goes like this: if the universe is finite, then what happens when you get to the edge of it and stick your hand through ... since this seems too bizarre to be possible, the universe must be infinite. But this argument is not sound because finite objects do not necessarily have boundary: for example, a circle is a one-dimensional object that is finite but has no boundary. It is harder to imagine three-dimensional spaces that are finite but have no boundary, but we will see several examples later on.

3 Gluing Diagrams

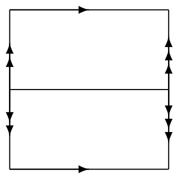
Next, I introduce gluing diagrams.

2. What do you get when you glue (or tape) the edges of this two-dimensional triangle together as shown? The matching arrows get glued together in the same direction.



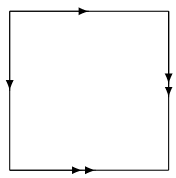
   Answer: a tetrahedron, which is topologically the same as a sphere.

3. What do you get when you glue together the edges of a square as shown?



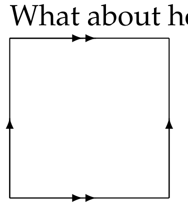
Answer: a sphere. Note that the inside of the square is not supposed to get glued to itself. Only the edges get glued up.

4. What about here?



Answer: a sphere again.

5. What about here?



Answer: a torus! When you glue along the first pair of arrows, you get a cylinder. With paper, is not physically possible to then glue along the second pair of arrows. But if you were using a very stretchy material instead of paper, you could glue together the second set of arrows by the attaching the ends of the cylinder to make a torus. A flexible tube is handy to demonstrate this. There is a subtle point here that it may not be necessary to discuss: this gluing diagram represents a surface that is topologically the same as the surface of a doughnut. However, the glued surface and the doughnut surface have different geometries: the gluing diagram is flat because it is made from a piece of flat paper, but the surface of a doughnut is curved.

6. What dimension is the torus? Is if finite or infinite? Does it have boundary?

Answer: two-dimensional, finite, no boundary.

4 The Torus

It's useful to think about surfaces as if we were creatures living in them, so that we don't have to visualize twisting them around in three dimensions. This will be good practice when we go up a dimension and try to understand three-dimensional spaces, since we can't imaging twisting these around in four-dimensional space.

7. If you were a two-dimensional creature living in the two-dimensional torus, drawn as a square with opposite sides attached as above, what would happen if you kept traveling up (towards the top of the square)? What if you kept traveling left?

You would come back from the bottom edge of the square; you would come back from the right side. There is a nice computer demonstration of this in Torus Games which you can download at http://geometrygames.org/TorusGames/index.html – select the Practice Board from the Help menu.

8. If you were a two-dimensional creature living inside the torus, with your eyes looking out the top of the square, what would you see if you looked forward? What if you looked right?

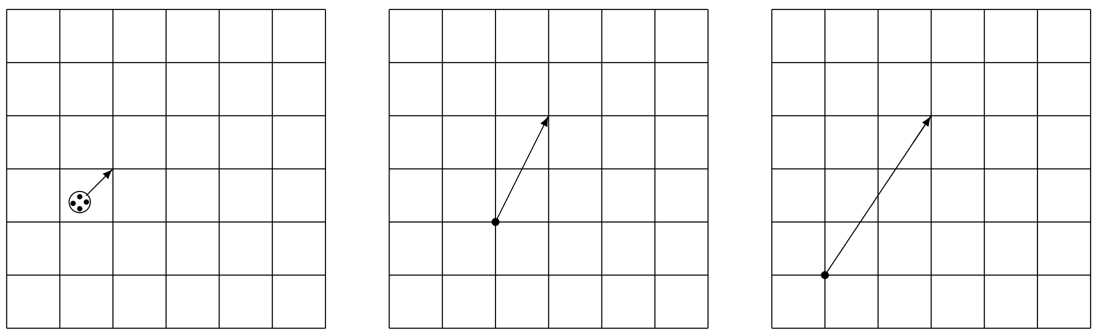
   The back of your head; your left side.

9. A ladybug on a torus walks in a straight line until she returns to her starting point. Draw her path and find the length of her path if:

   (a) she walks 1 unit northward for every 1 unit eastward, as shown at the left.

   (b) she walks 2 units northward for every 1 unit eastward as shown in the center.

   (c) she walks 3 units northward for every 2 units eastward as shown at right.

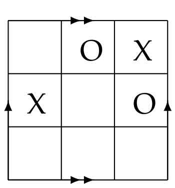


10. If the ladybug walks n units northward for every e units eastward, where n and e are integers, how long will her path be? What if n and e are not integers?

5 Tic Tac Toe on the Torus

I ask someone to explain how regular Tic Tac Toe works. On the torus, it's the same idea, but the opposite edges of the board are glued together. I do the first problem below, then play one game with a student, then let the students play each other in pairs. It's also fun to let the left side of the classroom play the right side, with a spokesperson from each side.

11. Where should X go to win? What if it's O's turn next, instead?



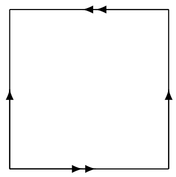
12. Is there a winning strategy for the first player in Tic Tac Toe on the torus? That is, is it possible for the first player to win no matter what the second player does? Does it change your answer if the first player is required to start on the upper left corner?

13. How many essentially different first moves are there in Tic Tac Toe on the torus? How many essentially different second moves are there?

14. A Cat's Game in Tic Tac Toe is a game where neither side wins, even though the board is filled up with X's and O's. Is it possible to have a Cat's Game in Tic Tac Toe on the torus?

6 The Klein Bottle

15. What do you get when you glue together the edges of this square as shown?



When you glue the left and right sides together, you get a cylinder. If you glue the top and bottom together, you get a Mobius band. This is a good opportunity for students to make Mobius bands and experiment with drawing a line down the center if they are not already familiar with Mobius bands.

In our three-dimensional world, it is not physically possible to glue the edges of the square in the cylinder direction and in the Mobius band direction simultaneously, even if our square is made of very stretchy rubber. If we glue in the cylinder direction to make a tube, the arrows on the ends of the tubes instruct us to attach them in a way that is impossible, unless we allow the tube to pass through itself. It's good to have a flexible tube to demonstrate with, and a model of a Klein bottle with self-intersection. It's important to realize that the the self-intersection is not an intrinsic part of the Klein bottle, but an artifact of trying to construct it in three dimensional space. If we had a fourth dimension to work with, we could lift the self-intersecting Klein bottle off itself to get one that doesn't intersect itself.

16. Imagine that you are a two-dimensional creature that lives in the Klein bottle, represented with the gluing diagram above. What happens if you keep walking left? What if you keep walking up?

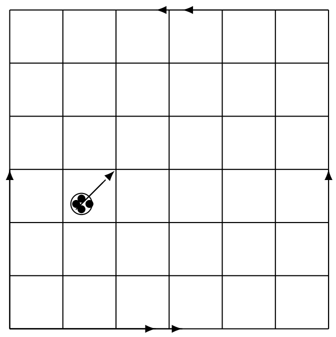
Answer: if you keep walking left, you go out the side of the square and return from the right. If you keep walking up, you return from the bottom, but your left arm returns on the right side and your right arm returns on the left. This is best demonstrated on the computer: if you download Torus Games from http://geometrygames.org/TorusGames/index.html , select "Klein bottle" from the Topology menu, and select the Practice Board from the Help menu, you'll get a nice way to try it out with asymmetric fish and other sea creatures. A possible alternative is to use a picture of an asymmetric creature drawn on a transparency. When it travels out the top of the square, it has to be flipped over to re-enter the bottom of the square in a way that agrees with the gluing instructions. If you write a word on the the creature, the word will end up backwards. The creature has turned into its mirror image.

A path that starts at the center of the square, goes up through the top, re-enters through the bottom and returns to the starting position is called an orientation-reversing path, because traveling around the path changes an object to its mirror image.

17. Can you find any other orientation-reversing paths in the Klein bottle?

There are many others. For example, you can go through the sides in addition to the top of the square, or you go through the top an odd number of times. A surface is called non-orientable if there is an orientation-reversing path in it. Otherwise, if there is no such path, it is called orientable.

18. A ladybug on a Klein bottle walks in a straight line until she returns to her starting point. She walks 1 unit northward for every 1 unit eastward. Draw her path.



19. Play Tic Tac Toe on the Klein Bottle with a friend. Is it possible to get a Cat's Game? Is there a winning strategy for the first player? How many essentially different first moves are there?

20. What do you get when you cut a Klein bottle in half? Depending on which way you cut it, how many different answers can you get?

    There is a nice video of this at http://www.youtube.com/watch?v=E8rifKlq5hc .

7 Three-dimensional spaces

So far we've been talking about two-dimensional creatures living in two-dimensional universes like the torus and the Klein bottle. The next step is to apply some of these same ideas to make a three-dimensional universe.

21. How can you make a three-dimensional universe analogous to the two-dimensional torus? This space is called a three-torus (not to be confused with a three-holed torus, which is a two-dimensional surface).

    Start with a cube shape, like the inside of the room, and glue the front wall to the back wall, the left wall to the right wall, and the ceiling to the floor.

22. If I go out of the front wall of a cube glued up to be a three-torus, where do I come back? (*The back wall.*) If I look up through the ceiling, what do I see? (*The bottom of people's feet.*)

23. Is a three-torus finite or infinite? (*finite*) Does it have boundary? (*no boundary, because all the walls are glued together in pairs*)

    Note that this is an example of a three dimensional universe that is finite but has no boundary, which may have been hard to imagine initially.

24. What if we glue the left wall to the right wall in the usual way, the ceiling to the floor in the usual way, but glue the front wall to the back wall with a flip? Now what happens if you go out the front wall? Do you feel any different? We'll call this space a three-dimensional Klein bottle.

    You come back mirror-reversed. However, you don't feel any different. From your perspective, you haven't changed but the whole room has become mirror reversed. Note: it is possible to specify the gluing instructions for gluing the front wall to the back wall by drawing any asymmetric figure identically on two sheets of paper and taping one to the front wall and one to the back wall.

25. Is the three-dimensional Klein bottle finite or infinite? Does it have boundary? Is it orientable? (*Finite, no boundary, non-orientable.*)

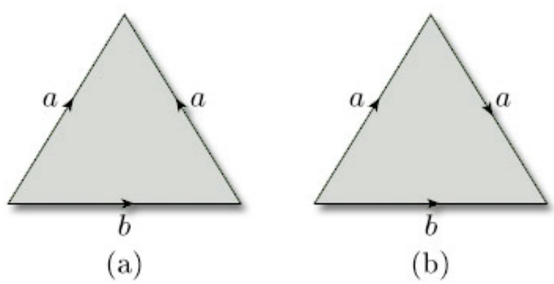
26. What are some good pranks you can play in a small, non-orientable universe?

My favorite is to take my friend's right shoe around an orientation reversing path while she is sleeping. What happens when she wakes up and tries to get dressed?

27. Suppose we live in a very large three-dimensional Klein bottle universe. Suppose some space explorers start at Earth and take a very long journey along an orientation reversing path. Finally they come to a planet that looks very familiar and they land on it. What happens when they exit the spacecraft and reach out their right hands to shake hands with the people that have come to greet them? What happens when they try to read the welcome signs that the people have made for them? What do the people on Earth think about the space travelers? If the space travelers have trouble adjusting to a mirror-reversed world, how can they fix their problem?

8 More gluing diagrams of surfaces

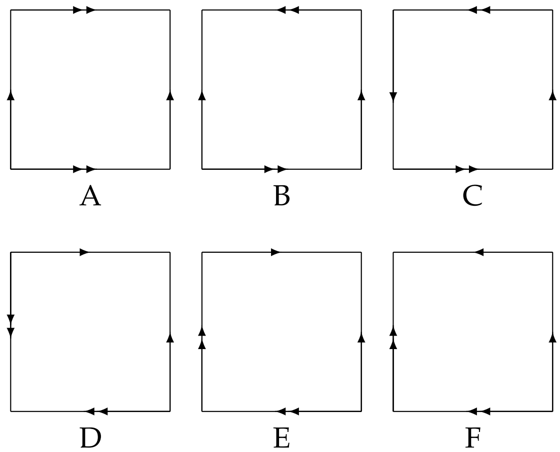
28. Which two surfaces are obtained by gluing the edges of each triangle as shown? You get two different surfaces, one for each triangle. Don't glue the two triangles together. Side b is not glued to anything.



(a) is a cone, which is topologically the same as a disk (i.e. a circle including the stuff inside it); (b) is a Mobius band. To see that (b) is a Mobius band, it is helpful to cut the entire right-hand figure along a vertical line down the center. Now you have two right triangles. Draw arrows labeled c to keep track of how the newly cut sides fit together. Now attach the two right triangles along the side a. This will require flipping one of the triangles over. The left and right edges of the resulting quadrilateral glue together along arrow c to form a Mobius band. This can all be physically done with poster board, scissors, and tape.

29. How many essentially different ways are their to glue the edges of a square in pairs? How many different topological surfaces result?

Let's call a pair of arrows a facing pair *if the two corresponding arrows point towards each other around the perimeter of the figure; otherwise, we'll call them a* chasing pair. *For example, diagram A has two facing pairs, while in diagram B, the single arrows form a facing pair but the double arrows form a chasing pair. One gluing diagram of a square can have either two chasing pairs of arrows, two facing pairs of arrows, or one chasing pair and one facing pair, so there are three options for this. In addition, there are essentially two options for which pairs of sides are glued together: either opposite sides are glued together or adjacent sides are glued together. This yields* $2 \times 3 = 6$ *essentially different gluing diagrams, which are drawn below.*



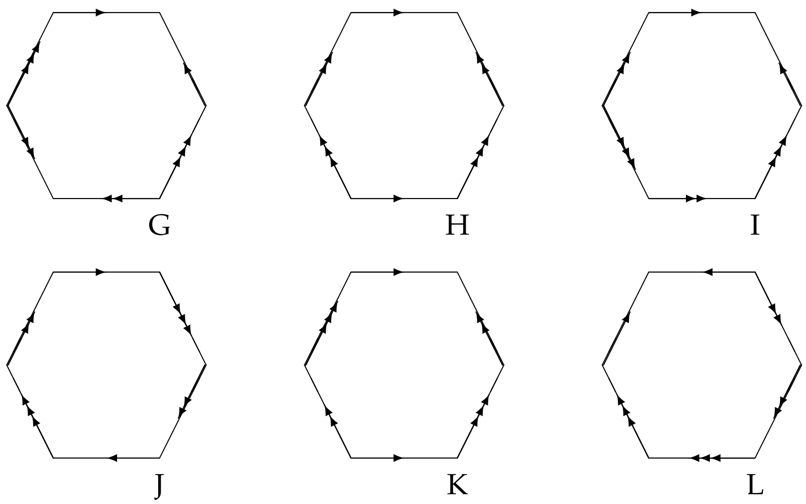
We've seen that A is a torus, B is a Klein bottle, and D is a sphere. Notice that any gluing diagram with a chasing pair of arrows produces a non-orientable surface. (Why?) So C, E, and F are all non-orientable. Are they all Klein bottles? Experiment by cutting down a diagonal and regluing edges to show that surface F is the same as surface B (a Klein bottle) and that surfaces C and E represent the same surface. Surfaces C and E are actually not *Klein bottles, and represent a different topological surface called a projective plane.*

30. How can we know for sure that the surface E is not topologically the same as surface B, the Klein bottle? We need some kind of invariant. For our purposes, an invariant of a topological surface is a number that we can calculate from a gluing diagram, but that really only depends on the topology of the surface represented by the diagram. So, two diagrams that represent the same surface should definitely get the same value of the invariant, and if possible, two diagrams that represent different surfaces should get different values, so that we can use the invariant to distinguish different surfaces. Try to find a number to assign to gluing diagrams so that both representations of the Klein bottle get the same number, and both representations of the projective plane get the same number, etc. Hint: look at what happens around the corners of the square in the gluing diagram.

    If a creature in the torus in diagram A starts walking in a small circle around the top left corner of the square, he ends up circling thorugh all four corners of the square in the gluing diagram. In other words, all four corners of the square in diagram A glue up around a single point, or vertex, on the completed torus surface. For each of the Klein bottles, B and F, there is also one vertex after gluing, but on D there are three, and on each of C and E there are two. So the number of vertices, together with the existence or non-existence of a chasing pair of arrows, distinguishes the surfaces for gluing diagrams involving squares. For hexagons, the situation is a bit more complicated.

9 Gluing Diagrams Using Hexagons

What surfaces do these gluing diagrams represent?



Answer: G = sphere, H = torus, I = torus, J = projective plane, K = Klein bottle , L = Klein bottle

31. For the six gluing diagrams using squares and the six gluing diagrams using hexagons drawn above, make a table listing the number of vertices after gluing, the number of edges after gluing, whether or not the surface is orientable, and the name of the surface (sphere, torus, etc.). Try to find a way to combine the number of edges after gluing and the number of vertices after gluing to get an invariant that is always the same number for two different gluing diagrams that represent the same surface.

    One possible invariant is $V - E$, where V is the number of vertices after gluing and E is the number of edges after gluing. The standard invariant used by topologists, called the Euler number, is something very similar: $V - E + F$, where F is the number of two-dimensional polygons in the gluing diagram. In all our examples, $F = 1$. It's a fact, that we will not prove here, that any two gluing diagrams that represent the same surface have the same Euler number. Amazingly, it is also true that if two gluing diagrams represent different surfaces, then either they have different Euler numbers, or else one is orientable and the other is non-orientable. So the Euler number, together with the orientability, completely determines the surface. (In this discussion, we are only talking about gluing diagrams that glue up all edges in pairs.)

32. What is the Euler number of a torus? A sphere? A Klein bottle? A projective plane?

    Answers: 0, 2, 0, 1

33. Can you find a gluing diagram of a hexagon that produces a new surface, that is not one of the ones we've studied so far?

    By reversing the direction of the arrow on the top of the hexagon L in the above figure, we can produce a surface with Euler number -1. Because its Euler number is -1, it cannot be a sphere, torus, Klein bottle, or projective plane.

34. How many other essentially different gluing diagrams can you make by identifying the sides of a hexagon in pairs? How many different topological surfaces can you get in this way?

10 Euler Number

$$\boxed{\chi = V - E + F}$$

35. There are three houses and three utilities (gas, water, and electricity). You must draw a line from each house to each utility, without the lines ever crossing. (The "lines" do not need to be straight lines.) Can you connect the houses to the utilities? What if the houses and the utilities and the lines connecting them all lie on a torus?

36. Can you position five points on a sheet of paper and connect each pair of points with a curved or straight line segment, so that none of the lines cross?What if the five points and the lines connecting them all lie on a torus?

37. Planet Casson is entirely dry land, except for a system of canals. One canal runs around the equator, and three canals run from three different spots on the equator to join up at the north pole. (So the canals form a shape like the edges of a tetrahedron.) The canals separate the land into 4 land masses. There is one ferry boat for each land mass that circles its land mass in a counterclockwise direction, traveling along the bounding canals. Unfortunately, the canals are too narrow for two ferries to pass each other, and there have been many crashes. Can you devise a ferry schedule with no crashes? (Ferries are not allowed to travel backwards.)

    What if you use a different arrangement of canals ... for example, an arrangement in the shape of the edges of a cube, or an octahedron?

    What if Planet Casson is shaped like a torus?

    Euler number can be used to show that these problems are impossible on a plane, or on the surface of a sphere, even though they are all possible on a torus!