East Lansing Elementary Math Circle
November 5, 2011
Towers of Hanoi
Audience: primarily 2nd-4th graders
Time: 90 minutes, but after preliminaries, about 75 minutes remaining for this lesson

Lesson Plan:

1. **Students explore the problem**

As students arrive, distribute handout (found on pages 2-3 of this document), along with pencils, and a set of coins (Quarter, Nickel, Penny, and Dime). Aim to seat them in pairs with a peer to work together.

Describe the problem and invite them to try to solve the game (moving the whole stack).

Often presented the problem like this:

"We're trying to move this stack of four coins to another circle. Let's see… *then just move the stack as a unit.* That doesn't seem like much of a challenge, does it? No, we have a few rules to make it a little trickier. One rule: We can only move one coin at a time, and we can only move the top coin from the stack. Another rule: We can't place a coin on top of a smaller coin." *Some students needed clarification to understand that they could put a dime on top of a quarter without the other coins in-between. You can put the coins on top of any coins that are bigger, not only the next biggest coin. The next most common misunderstanding was students who moved the coins around a good deal, but then wound up on the same circle.*

Have them work in groups/pairs to solve the game. Most students were able with few or no hints to solve the puzzle with four coins.

2. **Students record their results, look for patterns**

After showing a teacher, we asked them to record the sequence of moves and minimum number of moves to solve each size tower in the table in the handout. (Worked best with a partner—one to move, and one to record).

When a group completed that challenge, we gave them a dollar coin and suggested they try a stack of 5.

3. **Understanding the pattern in number of moves needed.**

After most groups had solved four and done some work on the table, we brought group together and make a chart with the minimum number of moves for each size of stack.

We spent a good deal of time asking about and identifying patterns which enable us to predict the next term. Different students observed:

a) the 2n+1 pattern (double and add one to get next term)

b) adding powers of two to get the next term

We used both methods to get the next terms, and pointed out that these are both "recursive" definitions and would take a lot of work to get to n=15. Any ideas about an "explicit" relationship? After some conversation, students did notice they were one less than powers of 2. Then, they could calculate the 15th term directly.

4. Solving the puzzle recursively

Next, we discussed a method for solving. We did this using the "Contractor game":

Equipment: Four rectangles of different colors and widths with magnets on back to represent the disks, labeled 1-4 (smallest is 1). Also, we had little signs with numbers 1-4, colored to match the rectangle, which each contractor held to identify his job.

Procedure: After drawing posts on the board, I arranged the smallest two rectangles to look like a (two disk) tower on one post. I asked someone to come move the two disk tower to another post. After a few practices, to make sure they knew how to move the tower to any other specified post, I asked for another volunteer. I added a third, larger rectangle form a three disk tower. I told the new volunteer that they are responsible for moving the whole stack to the third post. The bad news Is that they only have a robot arm which is capable of moving the third disk, and not any others. The good news is that they can subcontract to the 1-2 tower volunteer by asking them to move the 1-2 stack to any specified post. After a practice to two to confirm that they can move the whole 3 disk stack, I add a fourth disk, and ask for a new volunteer. The new volunteer again is told that they are responsible for moving the four disk tower, but that they only have the ability of move the 4 disk. Fortunately, they are able to contract to the three-tower mover. (They cannot talk directly to the 2 contractor, but the 3 contractor can.) By this point, they understood the recurrence relation, and we didn't proceed to the fifth disk, but we did talk about what the fifth "contractor" would have done.

We discussed the "recursive nature" of the process. What if we wanted to program a robot—would we have to write out the whole list of instructions? We also discussed how this recursive relationship related to the (2n+1) pattern in the number of moves required. The k-th contractor had to ask the (k-1)-contractor to do his/her routine twice, and the k disk had to be moved exactly once.

5. **Considering the pattern of moves in the solution.**

Finally, I asked them to read off the pattern of moves they had recorded in their table, and recorded them on the whiteboard with colors matching the colors of the disks we had used. By this time, it was easy to catch the pattern and the recursive nature of the pattern.

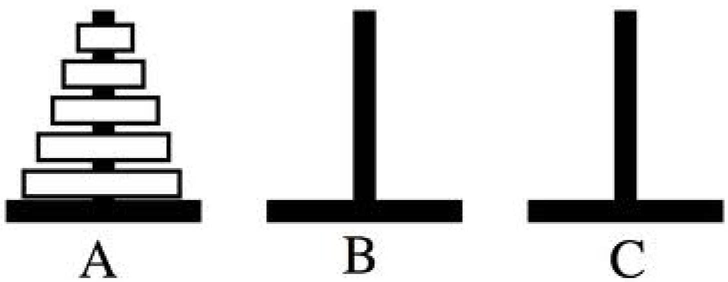
Lastly, I directed their attention to the ruler at the bottom of their page, and asked if they noticed any familiar patterns. I drew a ruler on the board with the markings color-coded to match the patterns we had used earlier (ie., one color for halves, others for fourths, eighths, and sixteenths). They immediately recognized this pattern. With a few more minutes, we could have used the ruler as a pattern to help us solve the puzzle directly.

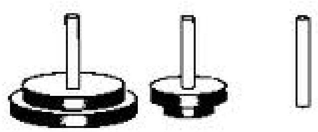
Towers of Hanoi Game

East Lansing Elementary Math Circle

Handout

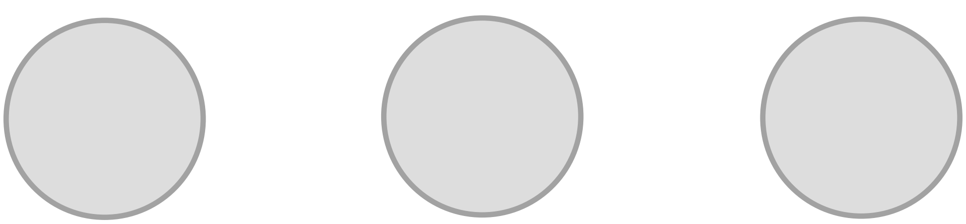
To play the Tower of Hanoi game, you need three pegs and a set of disks of different sizes. The goal is to move the entire tower to a new peg. You may only move one disk at a time, and you may not ever put a disk on top of a smaller disk. This is the starting position if you use five disks:



This is what you cannot do:  (look at the second peg)

Today, we can play the game with four coins: a quarter, nickel, penny, and dime (in that order). You can use the three circles below as your game board. You can also use a set of stacking cups, or a set of stacking disks.

Work with a partner to solve this puzzle. Record the sequence of coins you move to solve the puzzle. 1= dime, 2=penny, 3=nickel, 4=quarter. So, for example, “12” means that you moved the dime, and then the penny. On the back is a table you can use to record your results.



Number of disks	Minimum number of moves	Sequence of moves
1 (e.g., dime)		
2 (dime, penny)		
3 (D, P, N)		
4 (D, P, N, Q)		
5		
6		
15		

More questions to explore:

Do you notice any patterns, either in the number of moves needed, or in the sequences of moves?

What's your guess for how many moves it would take to solve a puzzle with 15 disks?

Assuming your guess is correct, if you moved one disk every second, how long would it take to solve?

Take a look at this ruler: If you label the shortest segments (16ths) A, the next (8ths) by B, and so on, what pattern do you get?