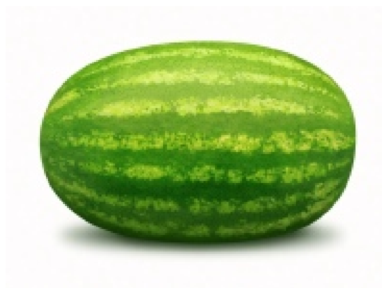
The Watermelon Problem
Slicing and Dicing with Lines and Planes

1 Slicing problems in various dimensions

1. What is the maximum number of pieces that you can cut a watermelon into with 4 straight cuts, if you aren't allowed to rearrange pieces between cuts? With 5 straight cuts?



2. What is the maximum number of pieces that you can cut a licorice stick into with 4 straight cuts? Again, you aren't allowed to rearrange pieces between cuts. With n straight cuts?

3. What is the maximum number of pieces that you can cut a pizza into with 4 straight cuts? With 5 straight cuts? With 6 straight cuts? As usual, you aren't allowed to rearrange pieces between cuts.

4. What if you are allowed to rearrange pieces between cuts?

5. Fill in as much of the following table as you can.

n # cuts	$L(n)$ max # of pieces of licorice	$P(n)$ max # pieces of pizza	$W(n)$ max # pieces of watermelon
0			
1			
2			
3			
4			
5			
6			

6. How are the numbers in the $P(n)$ column related to the numbers in the $L(n)$ column? Explain why this relationship holds. (Hint: look at the differences between consecutive numbers in the $P(n)$ column.)

7. Make a conjecture about the relationship between the numbers in the $W(n)$ column and the $P(n)$ column. Use your conjecture to fill in the rest of the table. Prove your conjecture – i.e., explain why the relationship holds.

8. Convince yourself that the maximum number of pieces that you can cut a pizza into with n straight cuts is the same as the maximum number of regions you can divide the plane into with n straight lines. Rephrase the watermelon problem in a similar way.

9. Generalize the watermelon problem to higher dimensions.

2 Formulas

10. Find a formula for $P(n)$ in terms of n. (Hint: $P(n)$ is a polynomial function of n. What is the degree of the polynomial? Use numbers from the table in part 5 to figure out the coefficients of the polynomial.)

11. Find a formula for $W(n)$ in terms of n. (Use the same hints as for the previous problem.)

12. Pascal's triangle is an infinite array of numbers:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
...

To construct Pascal's triangle, start with a 1 at the top and two 1's below it. For subsequent rows, to get each entry, add together the number just above it and to the left, and the number just above it and to the right. Put 1's along the left and right edges.

The pizza numbers $P(n)$ are lurking in Pascal's triangle. Find them. (Hint #1: you will need to add together some numbers in Pascal's triangle. Hint #2: if you number the rows starting with $n = 0$ for the top row, then the pizza number $P(n)$ can be found on the n^{th} row.)

13. Find the watermelon numbers $W(n)$ in Pascal's triangle.

14. Use your observations from parts 12 and 13 to write formulas for $P(n)$ and $W(n)$ in terms of binomial coefficients. Compare these formulas to your answers to parts 10 and 11. They should be equivalent!

    Note: The numbers in Pascal's triangle *are* the binomial coefficients. That is, if we number the rows of Pascal's triangle starting with $n = 0$ for the top row, and number the entries on each row starting with $k = 0$ for the leftmost entry, then the entry in row n and position k is $\binom{n}{k}$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-k+1)}{k!}$$

    For a proof of this non-obvious fact, see www.mast.queensu.ca/∼peter/inprocess/pascal.pdf or www.geometer.org/mathcircles/pascal.pdf.

15. Write down a general formula for the maximum number of pieces you can cut a k-dimensional watermelon into with n straight cuts in terms of binomial coefficients.

16. Use mathematical induction to prove your formulas in part 14. (Hint: use induction to prove the pizza formula first, using your observation from part 6 to justify the inductive step. Then use induction to prove the watermelon formula, using your established formula for $P(n)$ and your observation from part 7 in the inductive step.)

17. Use mathematical induction to prove the general formula in part 15.

3 More slicing and dicing problems

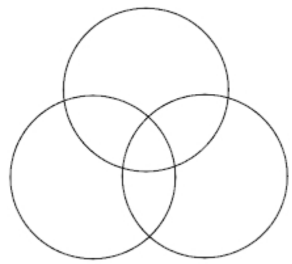
18. What is the minimum number of regions that you can divide space into with 4 distinct planes? The maximum number is the same as the answer to the original watermelon problem. Is it possible to get every number in between the minimum and the maximum? If so, demonstrate; if not, determine which numbers can be achieved and which cannot.

19. Suppose you draw 20 circles in the plane, all passing through the origin, but no two tangent at the origin. Also, except for the origin, no three circles pass through a common point. How many regions are created in the plane? [1]

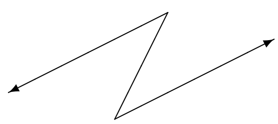
20. Take a sphere and draw a great circle on it (a great circle is a circle whose center is the center of the sphere). There are two regions created on the surface of the sphere. Now draw another great circle: there are four regions. Now draw a third, not passing through the points of intersection of the first two. How many regions are there? What is the maximum number of regions created by n great circles drawn on the surface of the sphere? [2]

[1,2] From Peter Taylor, www.mast.queensu.ca/∼peter/inprocess/cuttingplanes.pdf

21. A Venn diagram with two circles cuts the plane into 4 regions; a Venn diagram with 3 circles cuts the plane into 8 regions. Is it possible to draw a Venn diagram with 4 circles that cuts the plane into 16 regions? What if you use ovals instead of circles? What is the maximum number of regions that you can divide the plane into with n circles?



22. When you cut the plane with n "zig-zags" (two antiparallel rays joined by a line segment), how many regions can you divide the plane into? [3]



23. How many pieces, maximum, can you cut the crescent into with 5 straight cuts? [4]

[3]From Joshua Zucker, National Association of Math Circles Problem List: http://www.mathcircles.org/content/problem-list

[4]From Sam Lloyd, http://www.puzzles.com/puzzleplayground/Dissections.htm